Crepant resolutions and A-Hilbert schemes in dimension four

Sarah Elizabeth Davis

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Mathematics Department, University of Warwick

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Declaration

I declare that, to the best of my knowledge, the material contained in this thesis is original and my own work except where otherwise indicated, cited, or commonly known.

The material in this thesis is submitted for the degree of Ph.D. to the University of Warwick only, and has not been submitted to any other university.

Abstract

The aim of this thesis is to improve our understanding of when crepant resolutions exist in dimension four.

In three dimensions [BKR01] proved that for any finite subgroup $G \subset \mathrm{SL}(3, \mathbb{C})$ the G-Hilbert scheme G-Hilb$(\mathbb{C}^3)$ gives a crepant resolution of the quotient singularity $\mathbb{C}^3/G$.

In four dimensions very little is known about when crepant resolutions exist. In this thesis I present several approaches to this problem. I give an algorithm which determines, for quotients by cyclic subgroups of $\mathrm{SL}(4, \mathbb{C})$ whether or not a crepant resolution exists. This algorithm seeks to find a crepant resolution by performing a tree search.

In Chapter 4, building on the work of [CR02] in three dimensions, I calculate the A-Hilbert scheme for a family of abelian subgroups $A \subset SL(4, \mathbb{C})$. I show that this can be used to find a crepant resolution of $\mathbb{C}^4/A$.

Chapter 1

Introduction

In this thesis I discuss approaches to finding crepant resolutions for four dimensional abelian quotient singularities $\mathbb{C}^4/A$ for $A \subset \mathrm{SL}(4,\mathbb{C})$ a finite abelian group. In this chapter I discuss the background to the problem and review some definitions which I will use in later chapters. I give an overview of Craw and Reid's paper [CR02] on which the work of Chapter 4 is based.

1.1 Background

In 1978 McKay [McK80] observed a connection between the representation theory of finite subgroups $G \subset \mathrm{SL}(2,\mathbb{C})$ and the resolution of surface singularities arising from the quotient of $\mathbb{C}^2$ by the action of G. Namely, there is a one-to-one correspondence between the nontrivial irreducible representations of G and the exceptional prime divisors of the resolution $f : Y \to \mathbb{C}^2/G$.

McKay's observation was proved by Gonzalez-Springberg and Verdier [GSV83] and independently by Knörrer [Knö85]; both proofs use case analysis based on the classification of finite subgroups of $\mathrm{SL}(2,\mathbb{C})$.

Interest in a three dimensional version of the McKay correspondence started when Dixon, Harvey, Vafa and Witten introduced the orbifold Euler number [DHVW85], [DHVW86]. This motivated the work of Ito, Markushevich and Roan [Ito95a, Ito95b, BM94, MOP87, Mar97, Roa89, Roa94, Roa96] whose papers together give a case-by-case proof, using the classification of finite subgroups of $\mathrm{SL}(3,\mathbb{C})$, that there exist crepant resolutions $f : \tilde{X} \to X = \mathbb{C}^3/G$ such that the orbifold Euler numbers $\chi(\tilde{X}) = \chi(X)$.

In 1992 Reid, see [IR96], made the following conjecture

Conjecture 1.1.1. *Let $G \subset \mathrm{SL}(n,\mathbb{C})$ be a finite subgroup. $X = \mathbb{C}^n/G$ the quo-*

tient space and $f : Y \to X$ a crepant resolution. Then there exists a basis of $H^*(Y, \mathbb{Q})$ consisting of algebraic cycles in one-to-one correspondence with conjugacy classes of G.

Let R be any common multiple of the orders of all g in G, μ_R the group of complex Rth roots of unity and $\Gamma = \mathrm{Hom}(\mu_R, G)$. Ito and Reid [IR96] prove that there is a canonical one-to-one correspondence between junior conjugacy classes in Γ and crepant discrete valuations of X. They go on to prove the conjecture in the case $n = 3$. This gives a direct proof of the formula for the orbifold Euler number.

The introduction of the G-Hilbert scheme by Ito and Nakamura [IN96] provided a new way of finding resolutions. They prove that for finite $G \subset \mathrm{SL}(2, C)$ it is the minimal resolution of $\mathbb{C}^2/G$. Nakamura [Nak01] goes on to prove that for G a finite abelian subgroup of $\mathrm{SL}(3, \mathbb{C})$, a smooth crepant resolution of $\mathbb{C}^3/G$ is given by $\mathrm{Hilb}^G(\mathbb{C}^3)$. Craw and Reid [CR02] explain how to calculate A-Hilb$(\mathbb{C}^3)$ explicitly, where A denotes a finite abelian subgroup of $\mathrm{SL}(3, \mathbb{C})$. The same result is proved by Bridgeland, King and Reid [BKR01] for any (not necessarily abelian) subgroup of $\mathrm{SL}(3, \mathbb{C})$.

Ito and Nakajima [IN00] use the G-Hilbert scheme to give a general existence proof in two dimensions.

At the same time Dais et al. use toric geometry to investigate the existence problem:

Existence Problem: *For which $G \subset \mathrm{SL}(n, \mathbb{C})$ with $n \geq 4$ do there exist (projective) crepant resolutions of $\mathbb{C}^n/G$?*

They prove the existence of a crepant resolution for several families of examples: existence is proved for complete intersections of hypersurfaces [DHZ98], a necessary and sufficient condition is given in [DHH98] for cyclic quotient singularities of the the form $\frac{1}{r}(1, 1, \ldots, a, r - a - (n - 2))$ in $\mathrm{SL}(n, \mathbb{C})$, and [DHZ06] proves some necessary conditions on quotient singularities and gives more examples of cyclic quotient singularities. One of these necessary conditions is

Condition 1.1.2. Every point of age $n \geq 2$ must be expressible as the sum of n age 1 points.

Work of Firla and Ziegler [FZ99] shows that this condition is not sufficient. They give several examples of rational convex cones which do not admit a Hilbert partition: these cones are toric fans of quotient singularities, where non-admittance of a Hilbert partition is equivalent to non-existence of a crepant resolution.

1.2 Quotient singularities

We begin by stating some well known definitions.

Let $G \subset \mathrm{GL}(n,\mathbb{C})$ be a finite subgroup. Let $\mathbb{C}^n/G = \operatorname{Spec}\mathbb{C}[x_1,\dots,x_n]^G$ denote the variety given by the quotient of $\mathbb{C}^n$ by the action of G on it.

A *quasi-reflection* is an element $g \in \mathrm{GL}(n,\mathbb{C})$ of finite order such that $g - I_n$ has rank 1.

A finite subgroup $G \subset \mathrm{GL}(n,\mathbb{C})$ is *small* if it contains no quasi-reflections. A theorem of Chevalley and Shepard-Todd, [ST54], [Che55], means we need only consider small subgroups of $\mathrm{GL}(n,\mathbb{C})$.

A variety X is *Gorenstein* if it is Cohen–Macaulay and the canonical sheaf ω_X is invertible.

Proposition 1.2.1 ([Wat74])**.** *Let $G \subset \mathrm{GL}(n,\mathbb{C})$ be a small subgroup. Then $\mathbb{C}^n/G$ is Gorenstein if and only if $G \subset \mathrm{SL}(n,\mathbb{C})$.*

A variety X has a *resolution of singularities* if there exists a proper birational morphism $f\colon Y \to X$ such that Y is nonsingular.

Definition 1.2.2 ([Rei87])**.** A variety X has *canonical singularities* if it satisfies the following two conditions:

1. for some integer $r \geq 1$ the Weil divisor rK_X is Cartier.

2. if $f\colon Y \to X$ is a resolution of X and $\{E_i\}$ the family of all exceptional prime divisors of f, then

$$rK_Y = f^*(rK_X) + \sum_i a_i E_i, \text{ with } a_i \geq 0.$$

If every $a_i > 0$ then X is said to have *terminal singularities.*

$\sum a_i E_i$ is called the *discrepancy* of f. If all the $a_i = 0$ then f is called a *crepant resolution.*

For G a finite subgroup of $\mathrm{SL}(n,\mathbb{C})$, the quotient space $X = \mathbb{C}^n/G$ has canonical divisor $K_X = 0$. Under these conditions, if the resolution $f\colon Y \to X$ is crepant then $K_Y = f^*(K_X) = 0$.

Proposition 1.2.3 ([Rei80])**.** *Gorenstein quotient singularities are canonical.*

Theorem 1.2.4 ([Rei80]). *Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a finite group acting linearly on $\mathbb{C}^n$. Suppose G has no quasi-reflections, so that the map $\mathbb{C}^n \to \mathbb{C}^n/G = X$ is étale in codimension 1. Then X is canonical if and only if for every element $g \in G$ of order r, and ϵ any primitive root of 1, the diagonal form of the action of g is*

$$g\colon x_i \to \epsilon^{a_i} x_i$$

such that $0 \leq a_i < r$ with $\sum a_i \geq r$. We will denote such an element g by $\frac{1}{r}(a_1, \dots, a_n)$.

Remark 1.2.5. X is Gorenstein if and only if $\sum a_i \equiv 0 \mod r$.

From now on we will only consider abelian subgroups of $\mathrm{SL}(n, \mathbb{C})$. This means all our varieties are toric.

For a group $G = \left\langle \frac{1}{r}(a_1, \dots, a_n) \right\rangle$ we consider lattices of the form $L = \mathbb{Z}^n + \frac{1}{\mathrm{r}}(a_1, \dots a_n) \supset \mathbb{Z}^n$. Let $M = \mathrm{Hom}(L, \mathbb{Z})$ be the dual lattice of L. This is the lattice of invariant monomials.

A *strongly convex rational polyhedral cone* in $L_{\mathbb{R}}$ is a cone σ, with vertex at the origin, which is generated over $\mathbb{R}_{\geq 0}$ by a finite number of vectors of L.

If σ is a cone in L, the dual cone in M is the set

$$\sigma^{\vee} = \{m \in M_{\mathbb{R}} | \langle m, u \rangle \geq 0 \ \forall u \in \sigma\}.$$

Proposition 1.2.6 ([Ful93]). *An affine toric variety U_σ is nonsingular if and only if the cone σ is generated by part of a basis for the lattice L, in which case*

$$U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{(n-k)}, \ k = dim(\sigma).$$

A cone is called *nonsingular* if it is generated by part of a basis for the lattice.

Definition 1.2.7. [IR96] Let $L = \mathbb{Z}^n + \frac{1}{r}(a_1, \dots, a_n) \cdot \mathbb{Z}$ be a lattice. Define the *age* of a point $(b_1, \dots, b_n)$ of L to be

$$\sum_{i=1}^{n} b_i.$$

Since all of our groups are in $\mathrm{SL}(n, \mathbb{C})$ the age of every lattice point will be an integer. We call the points with age 1 *junior points.*

We denote by $\overline{a}$ the integer $a \mod r$, where r is the order of the group G, unless otherwise stated. The junior points of L are the points $\frac{1}{r}(\overline{ka_1}, \overline{ka_2}, \dots, \overline{ka_n})$, for $1 \leq k < r$, such that $\frac{1}{r}\sum_{i=1}^{n} \overline{ka_i} = 1$, together with the points

$$e_1 = (1, 0, \dots, 0), \quad e_2 = (0, 1, \dots, 0), \dots, \quad e_n = (0, 0, \dots, 1).$$

Let $G = \left\langle \frac{1}{r}(a_1, \dots, a_n) \right\rangle$. If $a_1 = 1$, for each $1 \leq b_1 < r$ there us a unique junior point in L with first coordinate b_1. Thus, we refer to the junior points $\frac{1}{r}(b_1, \dots, b_n)$ as p_{b_1}.

These points all lie on the plane $x_1 + \cdots + x_n = 1$. We refer to the intersection of this plane with the first orthant as the *junior simplex.* In dimension four this is a tetrahedron whose vertices are the points e_1, e_2, e_3, e_4.

Remark 1.2.8. In toric geometry it is well known that a crepant resolution $f \colon Y \to X$ is a toric fan of Y whose 1-skeleton consists only of junior points. See for example [Rei87]. That is, every cone is generated by part of the basis of the lattice (we say it is a *basic* cone) and this basis consists of rays generated by the junior points. Thus a crepant resolution of $X = \mathbb{C}^n/G$ is a triangulation of the junior simplex into r simplices of relative volume 1, where r is the order of the group G.

In the four dimensional case, if $p_1p_2p_3p_4$ is a simplex its volume is $\frac{1}{4!}$ times the determinant of the matrix $(p_{i,j})$ which is the volume of the parallelepiped with vertices p_1, p_2, p_3, p_4, where $p_{i,j}$ denotes the jth coordinate of p_i.

In toric geometry the addition of a ray through a point p in the interior of a cone corresponds to performing a blow-up at the point p.

Let G be the group generated by $\frac{1}{\mathrm{r}}(1, a)$ and let $L = \mathbb{Z}^2 + \frac{1}{\mathrm{r}}(1, a) \cdot \mathbb{Z}$ be a lattice. The Hirzebruch-Jung continued fraction $\frac{r}{a}$ is defined to be

$$\frac{r}{a} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cdots - \cfrac{1}{b_k}}}$$

We will use the notation $[b_1, b_2, \dots, b_k]$ for the Hirzebruch-Jung continued fraction of $\frac{r}{a}$.

For $f_i \in L$ we have

$$f_{i-1} + f_{i+1} = b_i f_i,$$

where $f_0 = \frac{1}{r}(0, r), f_1 = \frac{1}{\mathrm{r}}(1, a), f_{k+1} = \frac{1}{\mathrm{r}}(r, 0)$. The f_i generate rays which give the toric fan of the resolution of $\mathbb{C}^2/G$.

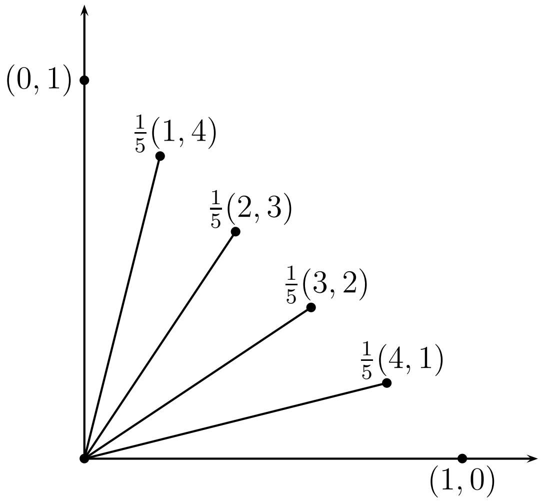

Figure 1.1: The Newton polygon for $\frac{1}{5}(1,4)$

Example 1.2.9. Let G be the group generated by $\frac{1}{5}(1,4)$ and L be the lattice $\mathbb{Z}^2 + \frac{1}{5}(1,4)\cdot\mathbb{Z}$. The Hirzebruch-Jung continued fraction is $\frac{5}{4} = [2,2,2,2]$ with

$$f_0 = (0,1), \quad f_1 = \frac{1}{5}(1,4), \quad f_2 = \frac{1}{5}(2,3), \quad f_3 = \frac{1}{5}(3,2)$$
$$f_4 = \frac{1}{5}(4,1), \quad f_5 = (1,0).$$

This gives the Newton polygon of Figure 1.1. Passing to the lattice M of invariant monomials we see that the dual cone of $\langle(1,4),(2,3)\rangle$ has basis $\alpha = x^4/y, \beta = y^2/x^3$, on which (1,4) and (2,3) are positive. The ideal $I = (x^4 = \alpha y, y^2 = \beta x^3)$ defines an affine piece of the resolution. The other four affine pieces can be calculated in the same way.

1.3 The G-Hilbert scheme

Let G be a finite subgroup of $\mathrm{SL}(n,\mathbb{C})$. A *$G$-cluster* is a G-invariant zero-dimensional subscheme $Z \subset \mathbb{C}^n$ with global sections $H^0(Z,\mathcal{O}_Z)$ isomorphic as a $\mathbb{C}[G]$-module to the regular representation of G. The *G-Hilbert scheme*, G-Hilb$(\mathbb{C}^n)$, is the moduli space of G-clusters.

Example 1.3.1. Let G be the group generated by

$$\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^4 \end{pmatrix},$$

with $\epsilon^5 = 1$. Since G acts on $\mathbb{C}^2$ by

$$\begin{aligned} x &\mapsto \epsilon x \\ y &\mapsto \epsilon^4 y \end{aligned}$$

this action leaves the monomials x^5, xy and y^5 invariant.

We wish to pick a monomial in each eigenspace of the group action. That is, for each $0 \leq i \leq 4$, a monomial u which is sent to $\epsilon^i u$. An obvious choice would be the monomials $\{1, x, x^2, x^3, x^4\}$.

$$\begin{array}{ll} y & \\ \boxed{1 \;\; x \;\; x^2 \; x^3 \; x^4} & x^5 \end{array}$$

Figure 1.2: A cluster of $\frac{1}{5}(1, 4)$

The remaining monomials of $\mathbb{C}[x, y]$ are in the ideal $\langle x^5, y\rangle$. We have relations $x^5 = \alpha$, $y = \beta x^4$ and $xy = \gamma$, with $\gamma = \alpha\beta$. Thus $\alpha = x^5$ and $\beta = y/x^4$ are local coordinates on a copy of $\mathbb{C}^2$. The cluster is illustrated in Figure 1.2.

The ideal $I = \langle x^5 = \alpha, y = \beta x^4\rangle$ defines a G-cluster:

$$Z = \mathrm{Spec}(\mathbb{C}[x, y]/\langle I\rangle)$$

with

$$H^0(Z, \mathcal{O}_z) = \mathbb{C}[x, y]/I.$$

This is isomorphic to the regular representation of G.

Other choices of relations give the different clusters:

$$\begin{array}{lll} x^4 = \alpha y, & y^2 = \beta x^3, & xy = \gamma \\ x^3 = \alpha y^2, & y^3 = \beta x^2, & xy = \gamma \\ x^2 = \alpha y^3, & y^4 = \beta x, & xy = \gamma \\ x = \alpha y^4, & y^5 = \beta, & xy = \gamma. \end{array}$$

Thus we have found five G-clusters, see Figure 1.3, which give us G-Hilb$(\mathbb{C}^2)$. These are exactly the dual cones of the cones of Example 1.2.9 shown in Figure 1.1.

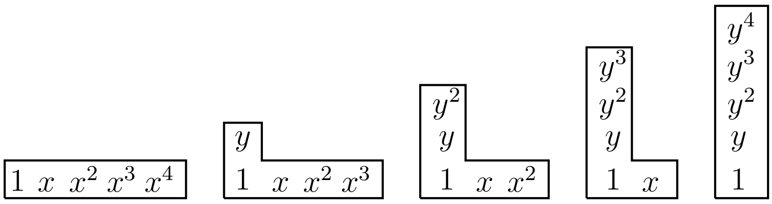

Figure 1.3: All clusters of $\frac{1}{5}(1,4)$

1.4 Calculating A-Hilb($\mathbb{C}^3$)

In 1999 Craw and Reid [CR02] gave an explicit construction of the G-Hilbert scheme for abelian subgroups of $\mathrm{SL}(3,\mathbb{C})$. This description of their construction follows [CR02] closely.

Let $A \subset \mathrm{SL}(3,\mathbb{C})$ be a finite abelian subgroup. A is generated by elements of the form $\frac{1}{r}(a_1, a_2, a_3)$ where $r = |A|$ and $0 \leq a_i < n$.

Let Δ be the junior simplex. $\mathbb{R}^2_\Delta$ is the plane spanned by Δ, and $\mathbb{Z}^2_\Delta = L \cap \mathbb{R}^2_\Delta$ is the corresponding lattice.

Definition 1.4.1. Write $\mathbb{Z}^2$ for the group of translations of the affine lattice $\mathbb{Z}^2_\Delta$. A *regular triple* is a set of three vectors $v_1, v_2, v_3 \in \mathbb{Z}^2$, any two of which form a basis of $\mathbb{Z}^2$, and such that $\pm v_1 \pm v_2 \pm v_3 = 0$.

Let $T \subset \mathbb{R}^2_\Delta$ be a triangle with vertices in $\mathbb{Z}^2_\Delta$ (so T is a *lattice triangle*). T is called a *regular triangle* if each of its sides is a line L_{ij} extending some $[e_i, f_{i,j}]$ and the 3 primitive vectors $v_1, v_2, v_3 \in \mathbb{Z}^2$ pointing along its sides form a regular triple.

A regular triangle T is affine equivalent to the triangle with vertices $(0,0), (r,0)$ and $(0,r)$ for some $r \geq 1$. We will call such a T a triangle of *side* r. A *regular tessellation* of T is the subdivision of T into r^2 basic triangles with sides parallel to the vectors $(r,0), (0,r)$ and $(-r,r)$. See Figure 1.4.

Craw-Reid use the Hirzebruch-Jung resolution at each of the vertices of Δ and an algorithm which contracts the concatenation of the Hirzebruch-Jung continued fractions at each vertex to split Δ into regular triangles which they then tessellate to give a triangulation of Δ.

Theorem 1.4.2. *The junior simplex, Δ, is partitioned by regular triangles.*

We now outline their proof.

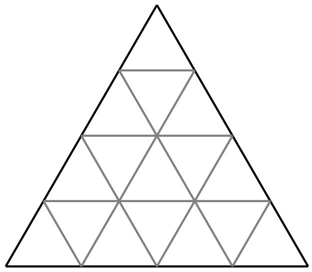

Figure 1.4: The regular triangulation of a triangle of side 4

At each e_i we construct the Newton polygon obtained as the convex hull of the lattice points in $\Delta \setminus e_i$. First we consider e_1 as the origin and x_2, x_3 as local coordinates. Thus the action of $\frac{1}{r}(a_1, a_2, a_3)$ becomes a $\frac{1}{r}(a_2, a_3)$ action. For a and r coprime we rewrite this in the form $\frac{1}{r}(1, b)$, which allows us to compute the Hirzebruch-Jung continued fraction

$$\frac{r}{b} = [b_{1,1}, b_{1,2}, \ldots, b_{1,k_1}].$$

We take the vectors $(0, r), (1, b)$ and $(r, 0)$ and use the continued fraction rule to find the remaining vectors $f_{1,j}$ of the Newton polygon:

$$f_{1,j-1} + f_{1,j+1} = b_{1,j} f_{1,j} \quad \text{for } j = 1, \ldots, k_1.$$

Here $f_{1,0}$ is the primitive vector along the side $[e_1, e_3]$ and f_{1,k_1+1} that along $[e_1, e_2]$. For the remaining e_i the corresponding vectors will be denoted $f_{i,0}, f_{i,1}, \ldots, f_{i,k_i+1}$ and are calculated in the same way.

Write $L_{i,j}$ for the line out of e_i extending or equal to the initial segment $[e_i, f_{i,j}]$. The resulting fan at e_i corresponds to the Hirzebruch-Jung resolution of the surface singularity $\mathbb{C}^2_{x_i=0}/A$.

Translating the Newton polygons at e_1, e_2 and e_3 to a common vertex gives a propellor shape. Note that $f_{i+1,0} = -f_{i,k_i+1}$, so multiplying all vectors in one of the propellor blades by -1 inverts that blade and gives a basic subdivision of a half-space. This enables us to express the vectors along the edges of Δ in terms of their neighbours:

$$c_{i+1} f_{i+1,0} = f_{i+1,1} - f_{i,k_i}$$

for some $c_{i+1} \in \mathbb{Z}$. If $c_{i+1} > 1$ the side $e_i e_{i+1}$ is called a *long side*. Thus we get a cyclic continued fraction

$$[c_1, b_{1,1}, b_{1,2}, \ldots, b_{1,k_1}, c_2, b_{2,1}, \ldots, b_{2,k_2}, c_3, b_{3,1}, \ldots, b_{3,k_3}] \tag{1.1}$$

where at least two of the c_i are equal to 1 by the following lemma.

Lemma 1.4.3. *The junior simplex Δ has at most one long side.*

If $c_1 = 1$ then $f_{1,0} = f_{1,1} - f_{3,k_3}$, so we can eliminate $f_{1,0}$:

$$(b_{1,1} - 1)f_{1,1} = f_{1,2} - f_{3,k}, \quad (b_{3,k_3} - 1)f_{3,k_3} = f_{3,k_3-1} - f_{1,1}.$$

This deletes the regular triangle with sides $f_{1,0}, f_{1,1}, f_{3,k_3}$, which is equivalent to the contraction of the 1 in the continued fraction:

$$a, 1, b \to a - 1, b - 1.$$

These contractions are continued until no more contractions are possible.

Lemma 1.4.4. *For brevity, call a chain of contractions taking a cyclic continued fraction (1.1) down to [1,1,1] an MMP.*

i. Every contraction of a 1 in an MMP corresponds to a regular triple.

ii. For every regular triple, there is an MMP ending at it.

iii. Every regular triple appears in every MMP.

This leads to an algorithm for calculating the subdivision into regular triangles. We first calculate the lines $L_{i,j}$ out of the vertices of Δ. Call the corresponding continued fraction entry, $b_{i,j}$, the *strength* of $L_{i,j}$. The lines $L_{i,j}$ are extended subject to the following rule. When two or more lines meet, the line with greater strength is extended, but its strength decreases by 1. Lines meeting with equal strength kill each other. This continues until all lines have been defeated. This partitions Δ into regular triangles. The final step is to take the regular tessellation of these regular triangles. Denote this by Σ.

Example 1.4.5. Let $A \subset \mathrm{SL}(3, \mathbb{C})$ be a finite subgroup generated by $\frac{1}{r}(a_1, a_2, a_3) = \frac{1}{13}(1, 2, 10)$. We consider the cyclic quotient singularity $\mathbb{C}^3/A$.

We obtain the singularity at e_i by setting $x_i = 0$, thus eliminating a_i. We use this to find the Hirzebruch–Jung continued fraction at each e_i:

At e_1: $\frac{1}{13}(2, 10) = \frac{1}{13}(1, 5)$ so we have $\frac{13}{5} = [3, 3, 2]$.

At e_2: $\frac{1}{13}(10, 1) = \frac{1}{13}(1, 4)$ so we have $\frac{13}{4} = [4, 2, 2, 2]$.

At e_3: $\frac{1}{13}(1, 2)$ so we have $\frac{13}{2} = [7, 2]$.

We may now compute the fans corresponding to the resolution of the singularities $\mathbb{C}^2_{x_i=0}/A$.

At e_1 we have $f_{1,0} + f_{1,2} = 3f_{1,1}$ and

$$f_{1,0} = (-13, 0, 13),\ f_{1,1} = (-6, 1, 5).$$

Thus

$$\begin{aligned} f_{1,2} &= 3f_{1,1} - f_{1,0} = (-5, 3, 2) \\ f_{1,3} &= 3f_{1,2} - f_{1,1} = (-9, 8, 1) \\ f_{1,4} &= 2f_{1,3} - f_{1,2} = (-13, 13, 0). \end{aligned}$$

Similarly at e_2 and e_3,

$$\begin{aligned} &f_{2,0} = (13, -13, 0), \quad f_{2,1} = (4, -5, 1), \qquad f_{2,2} = (3, -7, 4), \\ &f_{2,3} = (2, -9, 7), \qquad f_{2,4} = (1, -11, 10), \quad f_{2,5} = (0, -13, 13), \\ f_{3,0} = (0, 13, -13),\quad &f_{3,1} = (1, 2, -3),\quad f_{3,2} = (7, 1, -8),\quad f_{3,3} = (13, 0, -13). \end{aligned}$$

Figure 1.5 shows the lines $L_{i,j}$ corresponding to the $f_{i,j}$.

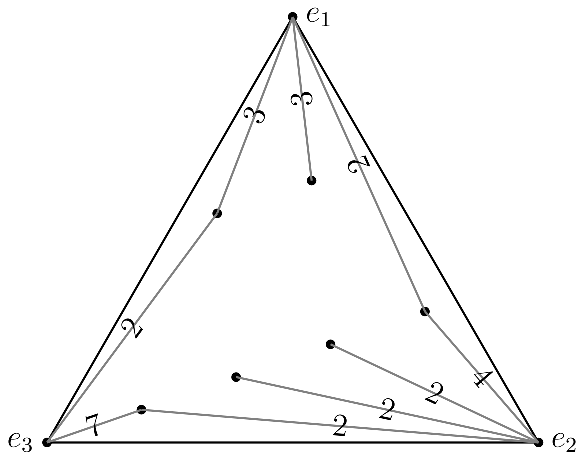

Figure 1.5: First step in obtaining a regular triangulation of $\frac{1}{13}(1, 2, 10)$.

Since the example is coprime, there are no long sides, so all $c_i = 1$. The concatenation of continued fractions is

$$[1, 3, 3, 2, 1, 4, 2, 2, 2, 1, 7, 2].$$

The second 1 denotes the regular triple $f_{2,0} = f_{2,1} - f_{1,3}$. Contracting this one corresponds to deleting the regular triangle $f_{2,0}, f_{2,1}, f_{1,3}$. The continued fraction becomes

$$[1, 3, 3, 1, 3, 2, 2, 2, 1, 7, 2].$$

Continuing this calculation (Lemma 1.4.4 says it doesn't matter which order we contract the 1s in) tells us how to extend the lines $L_{i,j}$ to give Figure 1.6. The dashed lines are the regular tessellation of the regular triangle with sides $L_{1,1}, L_{1,2}$ and $L_{3,1}$.

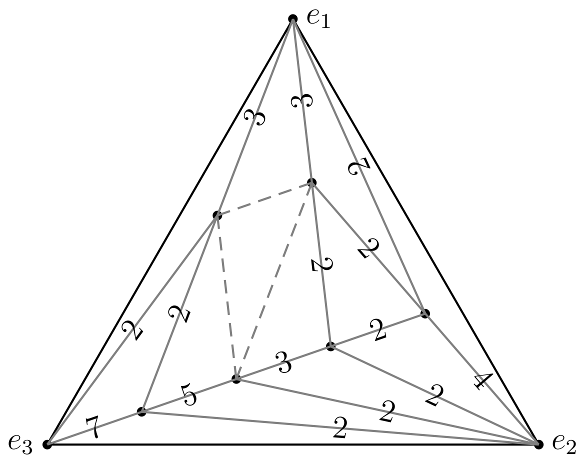

Figure 1.6: A regular triangulation of $\frac{1}{13}(1, 2, 10)$.

Theorem 1.4.6. *Let Σ denote the toric fan determined by the regular tessellation of all regular triangles in the junior simplex Δ. The toric variety X_Σ is Nakamura's A-Hilb$(\mathbb{C}^3)$.*

Corollary 1.4.7. *[Nak01] A-Hilb$(\mathbb{C}^3) \to \mathbb{C}^3/A$ is a crepant resolution.*

To prove Theorem 1.4.6, Craw and Reid show that passing to the dual basis of Σ in M, the lattice of invariant monomials, gives exactly the A-clusters of Nakamura's theorem:

Theorem 1.4.8. *(I). For every finite diagonal subgroup $A \subset \mathrm{SL}(3, \mathbb{C})$ and every A-cluster Z, generators of the ideal $\mathcal{I}_Z$ can be chosen as the system of 7 equations*

$$\begin{aligned} x^{l+1} &= \xi z^b t^f, & z^{b+1} t^{f+1} &= \lambda x^l, \\ z^{m+1} &= \eta x^d t^c, & x^{d+1} t^{c+1} &= \mu z^m, & \qquad xyzt = \pi, \\ t^{n+1} &= \zeta x^a z^e, & x^{a+1} z^{e+1} &= \nu t^n. \end{aligned} \tag{1.2}$$

Here $a, b, c, d, e, f, l, m, n \geq 0$ are integers, and $\xi, \eta, \zeta, \lambda, \mu, \nu, \pi \in \mathbb{C}$ are constants satisfying

$$\lambda\xi = \mu\eta = \nu\zeta = \pi.$$

(II). Moreover, exactly one of the following cases holds:

$$\text{“up”}\quad \begin{cases} \lambda = \eta\zeta, \mu = \zeta\xi, \nu = \xi\eta, \pi = \xi\eta\zeta \\ l = a + d, m = b + e, n = c + f; \end{cases}$$

or

$$\text{“down”}\quad \begin{cases} \xi = \mu\nu, \quad \eta = \nu\lambda, \quad \zeta = \lambda\mu, \quad \pi = \lambda\mu\nu, \\ l = a + d + 1, \quad m = b + e + 1, \quad n = c + f + 1. \end{cases}$$

Example 1.4.9 (Example 1.4.5 continued)**.** The regular triangle $f_{3,1} = \frac{1}{13}(1, 2, -3)$, $f_{1,2} = \frac{1}{13}(-5, 3, 2)$, $f_{2,1} = \frac{1}{13}(4, -5, 1)$ of side 1 has dual basis

$$\xi = x^2/y, \quad \eta = y^2/z^3, \quad \zeta = z^4/x$$

which gives equations $x^2 = \xi y, y^2 = \eta z^3, z^4 = \zeta x$. It is not hard to see that the other equations of (1.2) can be generated from these, and thus they define the ideal $\mathcal{I}_Z$ of the cluster Z.

The triangle $f_{1,1}, f_{1,2}, f_{3,1}$ is a regular triangle of side 2. The sides of the dual to the this triangle are cut out by

$$\xi = x^2/y, \quad \eta = y^5/z, \quad \zeta = z^3/y^2.$$

The regular tessellation is given by pushing in the sides of the triangles by i, j and k steps, for $0 \leq i, j, k \leq r - 1$ integers, respectively. The case $i + j + k = r - 1$ gives triangles which have the same orientation as the original triangle — they are referred to as “up” — and the cases $i + j + k = r + 1$ gives triangles which have the opposite orientation to the original triangle — they are “down” triangles. Pushing the first side in by i steps corresponds to multiplying ξ by $(xyz)^i$. Thus the four triangles of the regular tessellation have dual basis

$$\xi = x^2/y, \quad \eta = y^5/z, \quad \zeta = z^2/xy^3 \tag{1.3}$$
$$\xi = x^2/y, \quad \eta = y^4/xz^2, \quad \zeta = z^3/y^2$$
$$\xi = x/y^2z, \quad \eta = y^5/z, \quad \zeta = z^3/y^2$$
$$\xi = x/y^2z, \quad \eta = y^4/xz^2, \quad \zeta = z^2/xy^3. \tag{1.4}$$

The dual bases of the regular triangles are shown as ratios in Figure 1.7.

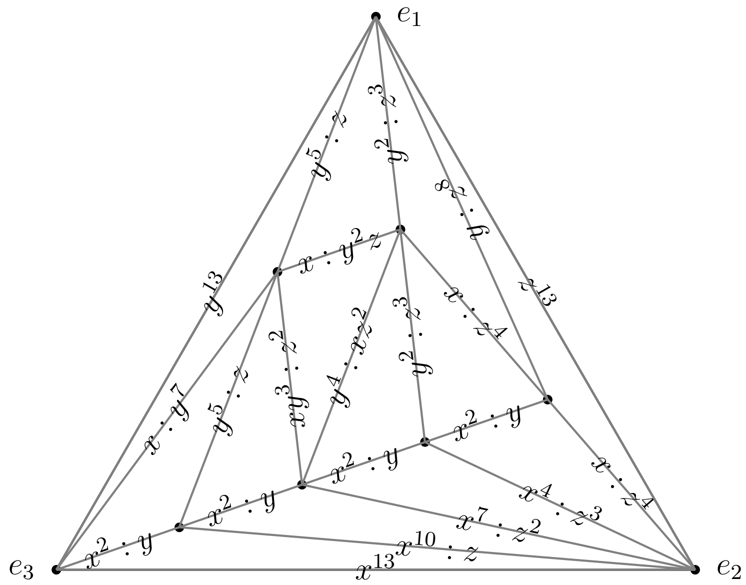

Figure 1.7: Ratios on the exception curves in A-Hilb$(\mathbb{C}^3)$ for $\frac{1}{13}(1,2,10)$.

The dual basis of (1.3) gives relations

$$x^2 = \xi y, \quad y^5 = \eta z, \quad z^2 = \zeta x y^3.$$

It is easy to see that these give rise to relations

$$y^2 z = \lambda x, \quad x z^2 = \mu y^4, \quad x^2 y^4 = \nu z$$

which satisfy the "up" case of Theorem 1.4.8.

The relations for (1.4) are

$$y^2 z = \lambda x, \quad x z^2 = \mu y^4, \quad x y^3 = \nu z^2.$$

These generate

$$x^2 = \xi y, \quad y^5 = \eta z, \quad z^3 = \zeta y^2.$$

which satisfy the "down" case of Theorem 1.4.8.

1.5 Hilbert partition definitions

Firla and Ziegler consider cones in a slightly different way. These definitions are taken from [FZ99].

A *pointed, rational, polyhedral cone* $C \subseteq \mathbb{R}^n$ is a set

$$\begin{aligned} C &= \operatorname{cone}\{a^1, \dots, a^m\} \\ &:= \{\lambda_1 a^1 + \cdots + \lambda_m a^m \in \mathbb{R}^n : \lambda_i \in \mathbb{R}, \lambda_i \geq 0 \text{ for } i = 1, \dots, m\} \end{aligned}$$

where $a^1, \dots, a^m \in \mathbb{R}^n$ are rational vectors.

A finite set of integer vectors $h^1, \dots, h^k$ is a *Hilbert basis* of C if each integral vector in C is a nonnegative integral combination of $\{h^1, \dots, h^k\}$.

Definition 1.5.1. Let $\mathcal{C} = \{C^1, \dots, C^r\}$ be a set of subcones of a cone C. We call a facet F of a subcone C^i an *interior facet* if $F \nsubseteq \delta C$, where δC denotes the boundary of C. We enumerate $\mathcal{F} := \{F^1, \dots, F^s\}$ the set of all interior facets of the cones in $\mathcal{C}$.

A point $g_0 \in \operatorname{int}(C)$ is called a *generic point* (with respect to $\mathcal{C}$) if it is not contained in the boundary of any of the subcones C^i.

Definition 1.5.2. Let C be a rational polyhedral pointed cone and $\mathcal{C} = \{C^1, \dots, C^r\}$ be a finite family of subcones.

The family $\mathcal{C}$ is a *cover* of C if every point of C is contained in one of the subcones C^i, that is, if $C = \cup_{i=1}^r C^i$.

$\mathcal{C}$ is a *binary cover* of C if

1. every generic point $g_0 \in C$ is contained in an odd number of subcones C^i, and
2. every interior facet F^j is a facet of an even number of subcones C^i.

A cover $\mathcal{C}$ is a *partition* if the intersection of any two subcones $C^i \cap C^j$ is a face of both cones, that is, if $\mathcal{C}$ forms a polyhedral complex.

$\mathcal{C}$ is a *regular partition* if additionally the complex is given by the domains of linearity of a piecewise linear convex function on C.

A cone $C \subseteq \mathbb{R}^n$ is *simplicial* if it is generated by a linearly independent set of vectors. A simplicial cone C is *unimodular* if it is generated by a subset of a basis of the lattice $\mathbb{Z}^n$, that is, if $C = \operatorname{cone}\{a^1, \dots, a^k\} \subseteq \mathbb{R}^n$ for some set $\{a^1, \dots a^n\}$ of integral vectors with $|\det\{a^1, \dots, a^n\}| = 1$.

Proposition 1.5.3. *(The Hilbert Cover Hierarchy) [FZ99, Proposition 3] Let $C \subseteq \mathbb{R}^n$ be an n-dimensional pointed rational polyhedra cone, and let $\mathcal{U} = \{C^1, \dots, C^s\}$*

be the (finite) set of all n-dimensional unimodular subcones of C that are generated by a subset of the Hilbert basis $\mathcal{H} = \mathcal{H}(C)$.

Each of the following properties of C implies the following ones:

Regular Hilbert Partition: *Some subset $\mathcal{C} \subseteq \mathcal{U}$ is a regular partition of C.*

Hilbert Partition: *Some subset $\mathcal{C} \subseteq \mathcal{U}$ is a partition of C.*

Binary Hilbert Cover: *Some subset $\mathcal{C} \subseteq \mathcal{U}$ is a binary cover of C.*

Hilbert Cover: *$\mathcal{U}$ is a cover of C.*

Integral Carathéodory Property: *Every integral vector $x \in C \cap \mathbb{Z}^n$ can be written as a nonnegative integral combination of at most n elements of the minimal Hilbert basis $\mathcal{H}(C)$.*

It is clear that the existence of a Hilbert partition is equivalent to the existence of a crepant resolution in the sense of Remark 1.2.8. Their Hilbert basis $\mathcal{H}$ corresponds to our set of junior points and their unimodular cones are exactly what we call basic cones.

Firla and Ziegler consider cones of the form

$$C[a_1, a_2, \ldots, a_n] := \operatorname{cone} \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ a_1 & a_2 & \ldots & a_{n-1} & a_n \end{pmatrix}$$

That is the cone spanned by the first $n-1$ unit vectors together with $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$.

In four dimensions, their aim of finding a partition of the cone $C[a_1, a_2, a_3, a_4]$ is equivalent to finding a triangulation of the the first orthant in a lattice of the form $L = \mathbb{Z}^4 + \frac{1}{r}(b_1, b_2, b_3, b_4)$. We now show how to translate between these two notations.

Since $L \cong \mathbb{Z}^5/(b_1, b_2, b_3, b_4, -r)$, there is a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{A} \mathbb{Z}^5 \xrightarrow{B} \mathbb{Z}^4 \to 0 \tag{1.5}$$

where the map A is given by

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ -r \end{pmatrix}$$

Computing the Smith normal form of A shows that the cokernel of A is $\mathbb{Z}^4$ and allows us to find the integer kernel K of A. Thus we have a short exact sequence

$$0 \to \mathbb{Z}^4 \xrightarrow{K} \mathbb{Z}^5 \xrightarrow{A^T} \mathbb{Z} \to 0$$

Dualising this shows that the map B in (1.5) is equal to the transpose of K. Now, for e_1, e_2, e_3, e_4, e_5 considered as the standard basis vectors of $\mathbb{Z}^5$, the vectors $Be_1, Be_2, Be_3, Be_4, B(b_1, b_2, b_3, b_4, -r)$ generate a cone in $\mathbb{Z}^4$. The last vector is generated by the rest, so we require only the first four. These can be mapped to the vectors generating $C[a_1, a_2, a_3, a_4]$ by an appropriate matrix of determinant one.

Example 1.5.4. Let $L = \mathbb{Z}^4 + \frac{1}{39}(1, 5, 8, 25) \cdot \mathbb{Z}$. Consider the cone generated by the standard basis vectors and the vector $\frac{1}{39}(1, 5, 8, 25)$.

The Smith normal form of

$$A = \begin{pmatrix} 1 \\ 5 \\ 8 \\ 25 \\ -39 \end{pmatrix}$$

is $UAV = (1, 0, 0, 0, 0)$, where $U = (1) \in \mathrm{GL}(1, \mathbb{Z})$ and $V \in \mathrm{GL}(5, \mathbb{Z})$ is

$$\begin{pmatrix} 1 & -5 & -8 & -25 & 39 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Now K is the last four columns of V, and B is the transpose of K:

$$B = \begin{pmatrix} -5 & 1 & 0 & 0 & 0 \\ -8 & 0 & 1 & 0 & 0 \\ -25 & 0 & 0 & 1 & 0 \\ 39 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Applying B to the vectors $e_1, e_2, e_3, e_4, \frac{1}{39}(1, 5, 8, 25)$ gives

$$(-5, -8, -25, 39), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1).$$

It is clear that $(0, 0, 0, 1)$ is generated by the rest, so we have the cone generated by

$$(-5, -8, -25, 39), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0).$$

We want a cone $C[a_1, a_2, a_3, a_4]$ with $a_i > 0$. The matrix

$$G = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

takes these vectors to

$$(14, 31, 34, 39), (0, 0, 1, 0), (0, 1, 0, 0), (1, 0, 0, 0),$$

which is the cone $C[14, 31, 34, 39]$ in the notation of [FZ99].

Chapter 2

Resolutions

Four dimensional Gorenstein quotient singularities do not always have a crepant resolution. Terminal singularities $\frac{1}{\mathrm{r}}(i, r-i, j, r-j)$ are obvious examples which cannot be resolved in this way. However there are also many non-terminal singularities for which no crepant resolution exists. This problem leads to many questions: For which singularities does a crepant resolution exist? Is there an invariant which obstructs the existence of a crepant resolution? Is there a higher dimensional analogue of crepant resolutions for higher dimensional singularities?

In this chapter several approaches to these questions are discussed.

2.1 Resolutions

A traditional way of obtaining a resolution was to perform blow-ups. In toric examples this is equivalent to barycentric subdivision: given a point p in a tetrahedral simplex $e_1e_2e_3e_4$ we cut along the plane segment pe_ie_j for each $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$ to obtain four smaller tetrahedral simplices:

$$pe_2e_3e_4, \quad e_1pe_3e_4, \quad e_1e_2pe_4, \quad e_1e_2e_3p.$$

In some cases a chain of such subdivisions will lead to a crepant resolution.

Example 2.1.1. Consider the quotient singularity $\frac{1}{17}(1, 3, 3, 10)$. This has five junior points

$$p_1 = \frac{1}{17}(1, 3, 3, 10), p_2 = \frac{1}{17}(2, 6, 6, 3), p_6 = \frac{1}{17}(6, 1, 1, 9),$$
$$p_7 = \frac{1}{17}(7, 4, 4, 2), p_{12} = \frac{1}{17}(12, 2, 2, 1).$$

Subdividing at p_1 gives

$$p_1e_2e_3e_4, \quad e_1p_1e_3e_4, \quad e_1e_2p_1e_4, \quad e_1e_2e_3p_1 \tag{2.1}$$

which have volume 1, 3, 3,10 respectively. Now, $p_2 = \frac{1}{10}(3p_1 + e_1 + 3e_2 + 3e_3)$ so is inside $e_1e_2e_3p_1$. Subdividing this simplex at p_2 yields:

$$p_2e_2e_3p_1, \quad e_1p_2e_3p_1, \quad e_1e_2p_2p_1, \quad e_1e_2e_3p_2 \tag{2.2}$$

which have volume 1,3,3,3 respectively. Since $p_6 = \frac{1}{3}(p_1 + e_1 + e_4)$, it is contained in the face $p_1e_1e_4$, and so the second and third simplices of (2.1) can each be subdivided at p_6 into three volume 1 simplices. Since p_7 and p_{12} lie on the line e_1p_2, the second, third and fourth simplices of (2.2) are each split into three volume 1 simplices by consecutive subdivision at p_7 and p_{12}. This procedure results in 17 simplices of volume 1:

$$\begin{gathered}
p_1e_2e_3e_4, \quad p_6p_1e_3e_4, \quad p_6e_2p_1e_4, \quad e_1p_6e_3e_4, \quad e_1e_2p_6e_4, \\
e_1p_1e_3p_6, \quad e_1e_2p_1p_6, \quad p_2e_2e_3p_1, \quad p_7p_2e_3p_1, \quad p_7e_2p_2p_1, \\
p_7e_2e_3p_2, \quad p_{12}p_7e_3p_1, \quad p_{12}e_2p_7p_1, \quad p_{12}e_2e_3p_7, \quad e_1p_{12}e_3p_1, \\
e_1e_2p_{12}p_1, \quad e_1e_2e_3p_{12}.
\end{gathered}$$

Considering these simplices as basic cones gives the toric fan of the crepant resolution of $\frac{1}{17}(1, 3, 3, 10)$.

Unfortunately, a chain of consecutive barycentric subdivision does not always lead to a resolution. In fact, as the following example demonstrates, the resolution obtained depends on the order of subdivision.

Example 2.1.2. Let $L = \mathbb{Z}^4 + \frac{1}{23}(1, 3, 4, 15)\mathbb{Z}$ be a lattice. There are three points of L in the interior of the junior simplex: $p_1 = \frac{1}{23}(1, 3, 4, 15), p_2 = \frac{1}{23}(2, 6, 8, 7)$ and $p_8 = \frac{1}{23}(8, 1, 9, 5)$. The points e_4, p_1 and p_2 are collinear.

Subdividing at p_2, then at p_1 and p_8 gives the simplices:

$$\begin{aligned}
&\Delta_1 = \Delta(p_1, e_2, e_3, e_4), && \Delta_2 = \Delta(p_1, e_2, e_3, p_2), && \Delta_3 = \Delta(e_1, p_8, e_3, e_4), \\
&\Delta_4 = \Delta(e_1, p_8, p_1, e_4), && \Delta_5 = \Delta(p_1, p_8, e_3, e_4), && \Delta_6 = \Delta(e_1, p_8, e_3, p_2), \\
&\Delta_7 = \Delta((e_1, p_8, p_1, p_2), && \Delta_8 = \Delta(p_1, p_8, e_3, p_2), && \Delta_9 = \Delta(e_1, e_2, p_1, e_4), \\
&\Delta_{10} = \Delta(e_1, e_2, p_1, p_2), && \Delta_{11} = \Delta(e_1, e_2, e_3, p_2).
\end{aligned}$$

Subdividing first at p_8 gives the simplices:

$$\begin{aligned}
\Delta'_1 &= \Delta'(p_1, e_2, e_3, e_4), & \Delta'_2 &= \Delta'(p_1, e_2, e_3, p_2), & \Delta'_3 &= \Delta'(p_8, p_1, e_3, e_4),\\
\Delta'_4 &= \Delta'(p_8, p_1, e_3, p_2), & \Delta'_5 &= \Delta'(p_8, p_2, p_1, e_4), & \Delta'_6 &= \Delta'(p_8, e_2, p_1, p_2),\\
\Delta'_7 &= \Delta'(p_8, e_2, e_3, p_2), & \Delta'_8 &= \Delta'(e_1, p_8, e_3, e_4), & \Delta'_9 &= \Delta'(e_1, e_2, p_8, e_4),\\
\Delta'_{10} &= \Delta'(e_1, e_2, e_3, p_8).
\end{aligned}$$

The volume of each simplex is calculated by taking the determinant of the matrix with the coordinates of each vertex as its rows. For Δ_9 the volume is

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 3 & 4 & 15 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 4,$$

so Δ_9 is singular. To calculate the type of singularity on Δ_9 note that

$$p_1 = \frac{1}{23}(e_1 + 3e_2 + 4e_3 + 15e_4),$$

so

$$e_3 = \frac{1}{4}(23p_1 - e_1 - 3e_2 - 15e_4).$$

Taking the coefficients modulo 4 gives

$$\frac{1}{4}(3p_1 + 3e_1 + e_2 + e_4).$$

Taking p_1, e_1, e_2, e_4 as a basis of L shows that Δ_9 has the terminal singularity $\frac{1}{4}(3,3,1,1)$. Permuting the coordinates gives $\frac{1}{4}(1,3,1,3)$. Hence

$$L(\Delta_9) = \mathbb{Z}^4 + \mathbb{Z} \cdot \frac{1}{4}(1,3,1,3).$$

Tables 2.1 and 2.2 show the singularities on the simplices obtained by each of these orders of subdivision.

The simplices of volume one are nonsingular, so the third column is left blank. Since subdivision has been performed at all junior points the singularities on the simplices with volume greater than one are all terminal. Each resolution has three singular cones, but different order in which subdivisions were performed has led to different singularities. Thus the singularities on the resolution are not invariants

Barycentric subdivision of the junior simplex in $L = \mathbb{Z}^4 + \frac{1}{23}(1,3,4,15) \cdot \mathbb{Z}$

Simplex	Volume	Singularity
Δ_1	1	
Δ_2	1	
Δ_3	1	
Δ_4	1	
Δ_5	1	
Δ_6	1	
Δ_7	1	
Δ_8	1	
Δ_9	4	$\frac{1}{4}(1,3,1,3)$
Δ_{10}	4	$\frac{1}{4}(1,3,1,3)$
Δ_{11}	7	$\frac{1}{7}(2,5,1,6)$

Table 2.1: Subdivision of the junior simplex at p_2 first

Simplex	Volume	Singularity
Δ'_1	1	
Δ'_2	1	
Δ'_3	1	
Δ'_4	1	
Δ'_5	1	
Δ'_6	1	
Δ'_7	2	$\frac{1}{2}(1,1,1,1)$
Δ'_8	1	
Δ'_9	9	$\frac{1}{9}(5,4,1,8)$
Δ'_{10}	5	$\frac{1}{5}(3,2,4,1)$

Table 2.2: Subdivision of the junior simplex at p_8 first

of the resolution.

2.2 Unavoidable points

In Example 2.1.2, it was shown that different resolutions may be obtained by different orders of subdivision. If a simplex has volume greater than one then there are lattice points in the interior of the simplex. In Example 2.1.2, these lattice points must have age greater than one, since the junior simplex was subdivided at every age one point.

The first subdivision of Example 2.1.2 gave a resolution which was covered by eleven pieces, three of which were singular, having singularities $\frac{1}{4}(1,3,1,3)$, $\frac{1}{4}(1,3,1,3)$, $\frac{1}{7}(2,5,1,6)$.

The singularity $\frac{1}{4}(3,1,3,1)$ was on the piece Δ_9, which has coordinates in terms of e_1, e_2, p_1, e_4. We have

$$\begin{aligned}\frac{1}{4}(3p_1 + 3e_1 + e_2 + e_4) &= \frac{1}{4}\left(\frac{1}{23}(3,9,12,45)\right) + \frac{1}{4}\left(\frac{1}{23}(69,0,0,0)\right) \\ &\quad + \frac{1}{4}\left(\frac{1}{23}(0,23,0,0)\right) + \frac{1}{4}\left(\frac{1}{23}(0,0,0,23)\right) \\ &= \frac{1}{4}\left(\frac{1}{23}(72,32,12,68)\right) \\ &= \frac{1}{23}(18,8,3,17) \\ &= p_{18}.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{1}{4}(2p_1 + 2e_1 + 2e_2 + 2e_4) &= p_{12}, \\ \frac{1}{4}(p_1 + e_1 + 3e_2 + 3e_4) &= p_6.\end{aligned}$$

Thus these age 2 points must appear on this resolution of $\frac{1}{23}(1,3,4,15)$.

The results of this calculation and the analogous calculation for the second subdivision are displayed in Table 2.3. The first column contains the points $\frac{1}{23}(\overline{n1}, \overline{n3}, \overline{n4}, \overline{n15})$ and any others which were shown to appear on the resolution from this calculation. A ✓ in the third or fourth column indicates this point belongs to a singular cone.

The age 1 points must appear on the resolution, however they do not arise from the singularities on the affine pieces. This calculation will not say whether any age 3 points appear on the resolution, since the singularities being considered are terminal, and these will only show the age 2 points.

The table shows that in this example, any age 2 points which are the sum of age 1 points does not necessarily appear as a divisor on the resolution, whereas any points not of this form must appear on the resolution; they are *unavoidable*. This leads to the following:

Lemma 2.2.1. *If a resolution is crepant every age two point must be the sum of two age one points.*

Since this a necessary condition for the existence of a crepant resolution we will call the condition

"Every every age two point must be the sum of two age one points"

Point	Age	Subdivision at p_2 first	Subdivision at p_8 first
$\frac{1}{23}(1,3,4,15)$	1	p_1	p_1
$\frac{1}{23}(2,6,8,7)$	1	p_2	p_2
$\frac{1}{23}(3,9,12,22)$	2	$p_3 = p_1 + p_2$	$p_3 = p_1 + p_2$
$\frac{1}{23}(4,12,16,14)$	2	$p_4 = 2p_2$	$p_4 = 2p_2$
$\frac{1}{23}(5,15,20,6)$	2	✓	✓
$\frac{1}{23}(6,18,1,21)$	2	✓	✓
$\frac{1}{23}(7,21,5,13)$	2	✓	✓
$\frac{1}{23}(8,1,9,5)$	1	p_8	p_8
$\frac{1}{23}(9,4,13,20)$	2	$p_9 = p_1 + p_8$	$p_9 = p_1 + p_8$
$\frac{1}{23}(10,7,17,12)$	2	$p_{10} = p_2 + p_8$	$p_{10} = p_2 + p_8$
$\frac{1}{23}(11,12,21,19)$	2	✓	✓
$\frac{1}{23}(12,13,2,19)$	2	✓	✓
$\frac{1}{23}(13,16,6,11)$	2	✓	✓
$\frac{1}{23}(14,19,10,3)$	2	✓	✓
$\frac{1}{23}(15,22,14,18)$	3	age 3	age 3
$\frac{1}{23}(16,2,18,10)$	2	$p_{16} = p_1 + p_2$	$p_{16} = p_1 + p_2$
$\frac{1}{23}(17,5,22,2)$	2	✓	✓
$\frac{1}{23}(18,8,3,17)$	2	✓	✓
$\frac{1}{23}(19,11,7,9)$	2	✓	✓
$\frac{1}{23}(20,14,11,1)$	2	✓	✓
$\frac{1}{23}(21,17,15,16)$	3	age 3	age 3
$\frac{1}{23}(22,20,19,8)$	3	age 3	age 3
$\frac{1}{23}(24,3,4,15)$	2	$p_1 + e_1$	✓
$\frac{1}{23}(26,6,8,7)$	2	$p_2 + e_1$	✓
$\frac{1}{23}(8,24,9,5)$	2	✓	$p_8 + e_2$

Table 2.3: Points on the subdivisions of the junior simplex of $\frac{1}{23}(1,3,4,15)$

the *junior necessity condition* or *JunNec.*

Proof of 2.2.1. This is well known. If Y is a crepant resolution of X then the toric fan of Y consists only of basic cones. If p is an age two point it must lie in a cone of Y, but then it must be a $\mathbb{Z}$-linear combination of the generators of the cone. Since the cone is basic these generators correspond exactly to the age one points of the lattice. □

It is clear that every point of age greater than one must be expressible as the sum of age one points.

Example 2.2.2. A crepant resolution of the quotient singularity $\frac{1}{17}(1,3,3,10)$

was computed in Example 2.1.1. Its junior points were

$$p_1 = \frac{1}{17}(1,3,3,10),\ p_2 = \frac{1}{17}(2,6,6,3),\ p_6 = \frac{1}{17}(6,1,1,9),$$
$$p_7 = \frac{1}{17}(7,4,4,2),\ p_{12} = \frac{1}{17}(12,2,2,1).$$

This singularity has a crepant resolution, and all of the age two points of the form $\frac{1}{17}(\overline{n1}, \overline{n3}, \overline{n3}, \overline{n10})$ are the sum of two of the p_i:

Point	As sum of juniors
$\frac{1}{17}(1,3,3,10)$	p_1
$\frac{1}{17}(2,6,6,3)$	p_2
$\frac{1}{17}(3,9,9,13)$	$p_1 + p_2$
$\frac{1}{17}(4,12,12,6)$	$p_2 + p_2$
$\frac{1}{17}(5,15,15,16)$	$p_1 + p_2 + p_2$
$\frac{1}{17}(6,1,1,9)$	p_6
$\frac{1}{17}(7,4,4,2)$	p_7
$\frac{1}{17}(8,7,7,12)$	$p_2 + p_6$
$\frac{1}{17}(9,10,10,5)$	$p_2 + p_7$
$\frac{1}{17}(10,13,13,15)$	$p_1 + p_2 + p_7$
$\frac{1}{17}(11,16,16,8)$	$p_2 + p_2 + p_7$
$\frac{1}{17}(12,2,2,1)$	p_{12}
$\frac{1}{17}(13,5,5,11)$	$p_1 + p_{12}$
$\frac{1}{17}(14,8,8,4)$	$p_2 + p_{12}$
$\frac{1}{17}(15,11,11,14)$	$p_1 + p_2 + p_{12}$
$\frac{1}{17}(16,14,14,7)$	$p_2 + p_2 + p_{12}$

If the converse to Claim 2.2.1 was true it would provide an easy criterion for finding crepant resolutions. Unfortunately this is not the case.

2.3 Counter-examples to sufficiency of JunNec

Example 2.3.1. Let A be the group generated by $\frac{1}{39}(1,5,25,8)$ with corresponding lattice $L = \mathbb{Z}^4 + \frac{1}{39}(1,5,25,8)\mathbb{Z}$.

There are eight points of L in the interior of the junior simplex:

$$\begin{array}{lll} p_1 = \frac{1}{39}(1,5,25,8), & p_2 = \frac{1}{39}(2,10,11,16), & p_5 = \frac{1}{39}(5,25,1,8), \\ p_8 = \frac{1}{39}(8,1,25,5), & p_{10} = \frac{1}{39}(10,11,2,16), & p_{11} = \frac{1}{39}(11,16,10,2), \\ p_{16} = \frac{1}{39}(16,2,11,10), & p_{25} = \frac{1}{39}(25,8,5,1). & \end{array}$$

Claim 2.3.2. *The quotient singularity* $\frac{1}{39}(1,5,25,8)$ *satisfies JunNec, but does not admit a crepant resolution.*

This can be tested using the Magma code described in Chapter 3, see section 3.6 for details.

This example was also considered by Robert Firla and Günter Ziegler in [FZ99]. The cone $C[14,31,34,39]$ of [FZ99, Example10] is exactly the first orthant of $L = \mathbb{Z}^4 + \frac{1}{39}(1,5,25,8)\mathbb{Z}$, as described in Example 1.5.4. They show that it admits no Hilbert partition; existence of a Hilbert partition is equivalent to existence of a crepant resolution.

Nine more examples of cones with $r \leq 100$ (in the lattice notation) which satisfy JunNec but admit no Hilbert partition are given in [Fir97, §4.2]. Considering these as cones in a lattice $L = \mathbb{Z}^4 + \frac{1}{\mathrm{r}}(a_1,a_2,a_3,a_4)$ shows that the cones $C[15,43,51,54]$ and $C[21,39,49,54]$ correspond to the quotient singularities $\frac{1}{54}(1,3,11,39)$ and $\frac{1}{54}(1,5,15,33)$ respectively. These are the same singularity up to change of coordinates.

It is somewhat surprising that these are the only eight examples with $r \leq 100$ for which JunNec holds, but for which there is no crepant resolutions.

2.4 Products of singularities

Firla and Ziegler observe that all except the smallest of their examples - that is all except $\frac{1}{39}(1,5,8,25)$ - do not have unimodular facets. That is, there are junior points of the form $\frac{1}{r}(a,b,c,0)$, up to permuting coordinates. This is a consequence of one of the a_is not being coprime to r.

Observation: All the four dimensional simplicial cones found by Firla and Ziegler to have a binary Hilbert cover but no Hilbert partition come from groups whose order is not prime.

It is possible to express each group as the product of groups of lower order:

Notation 1	Notation 2	Product group
$\frac{1}{39}(1,5,8,25)$	$(14,31,34,39)$	$\frac{1}{3}(1,2,2,1)\times\frac{1}{13}(1,5,8,12)$
$\frac{1}{54}(1,3,11,39)$	$(15,43,51,54)$	$\frac{1}{2}(1,1,1,1)\times\frac{1}{27}(14,15,19,6)$
$\frac{1}{78}(1,5,20,52)$	$(26,58,73,78)$	$\frac{1}{2}(1,1,0,0)\times\frac{1}{3}(2,1,1,2)\times\frac{1}{13}(11,3,12,0)$
$\frac{1}{88}(1,11,32,44)$	$(44,56,77,88)$	$\frac{1}{11}(7,0,4,0)\times\frac{1}{8}(3,1,0,4)$
$\frac{1}{90}(1,9,35,45))$	$(45,55,81,90)$	$\frac{1}{2}(1,1,1,1)\times\frac{1}{5}(2,3,0,0)\times\frac{1}{9}(1,0,8,0)$
$\frac{1}{96}(1,8,39,48)$	$(48,57,88,96)$	$\frac{1}{3}(1,2,0,0)\times\frac{1}{32}(1,8,7,16)$
$\frac{1}{96}(1,11,36,48)$	$(48,60,85,96)$	$\frac{1}{3}(1,2,0,0)\times\frac{1}{32}(1,11,4,16)$
$\frac{1}{96}(1,15,32,48)$	$(48,64,91,96)$	$\frac{1}{3}(1,0,2,0)\times\frac{1}{32}(1,15,0,16)$

This observation is insufficient to explain the non-existence of a crepant resolution. The product of two terminal singularities need not be a terminal singularity, and may have a crepant resolution. The product

$$\frac{1}{3}(1,2,1,2)\times\frac{1}{5}(1,2,4,3)=\frac{1}{15}(8,1,2,4) \tag{2.3}$$

is not terminal and does have a crepant resolution. The cones

$$\begin{array}{ccccc}
p_2e_2e_3e_4, & p_4p_2e_2e_3, & p_8p_2e_2e_4, & p_8p_4p_2e_2, & p_1p_2e_3e_4,\\
p_1p_4p_2e_3, & p_1p_8p_2e_4, & p_1p_8p_4p_2, & e_1p_4e_2e_3, & e_1p_8e_2e_4,\\
e_1p_8p_4e_2, & e_1p_1e_3e_4, & e_1p_1p_4e_3, & e_1p_1p_8e_4, & e_1p_1p_8p_4,
\end{array}$$

which were produced by the program described in Chapter 3, form a crepant resolution.

However permuting the coordinates of the second group in the product (2.3) gives $\frac{1}{3}(1,2,1,2)\times\frac{1}{5}(1,4,2,3)=\frac{1}{15}(8,7,4,11)$, which is terminal and therefore can not have a crepant resolution.

It would be interesting to know whether or not this observation is connected to the non-existence of a crepant resolution in each of these cases.

2.5 The search for a sufficient condition

Since the examples of Firla and Ziegler do not have a crepant resolution they cannot be resolved via a chain of barycentric subdivisions. Thus somewhere JunNec must fail after a barycentric subdivision.

Condition 2.5.1. There exists a junior point p of the junior simplex, subdividing at which preserves the junior necessity condition.

This condition requires that after subdivision at the point p, the singularities on the new cones all satisfy the junior necessity condition.

The smallest example $\frac{1}{39}(1,5,25,8)$ does not satisfy Condition 2.5.1. Consider the junior simplex in the lattice $L = \mathbb{Z}^4 + \mathbb{Z} \cdot \frac{1}{39}(1,5,8,25)$. Subdivision at p_1 gives the four simplices

$$p_1e_2e_3e_4, \quad e_1p_1e_3e_4, \quad e_1e_2p_1e_4, \quad e_1e_2e_3p_1.$$

The first of these has relative volume 1, and so is nonsingular, but the others have the singularities $\frac{1}{5}(4,4,2,0), \frac{1}{8}(7,7,3,7)$ and $\frac{1}{25}(14,24,20,17)$ respectively. Now consider the lattices generated by each of these singularities. In the $\frac{1}{5}(4,4,2,0)$ and $\frac{1}{25}(12,24,20,17)$ cases, every age 2 point is the sum of two age 1 points. However this is not true for $\frac{1}{8}(7,7,3,7)$. The age 1 points are

$$\frac{1}{8}(1,1,5,1), \quad \frac{1}{8}(2,2,2,2),$$

and there is no way to make $\frac{1}{8}(5,5,1,5)$ as a $\mathbb{Z}$-linear combination of these.

The order of subdivision does not matter here. Subdividing the junior simplex at any junior point leads to a simplex whose singularity does not satisfy JunNec.

This led to the following conjecture:

Conjecture 2.5.2. *There exists a crepant resolution if and only if every age two point is the sum of two age one points and Condition 2.5.1 is satisfied.*

However, this conjecture is false. This condition is satisfied by $\frac{1}{54}(1,3,11,39)$, however this example does not have a crepant resolution, so cannot have a complete chain of barycentric subdivisions. Subdivision at the points $\frac{1}{54}(18,0,36,0)$ and $\frac{1}{54}(36,0,18,0)$ preserves JunNec, but it is not preserved by subsequent subdivisions.

On the other hand, the junior simplex of the lattice $L = \mathbb{Z}^4 + \frac{1}{67}(1,5,8,53)\mathbb{Z}$, also does not satisfy the condition, and no order of barycentric subdivisions leads to a crepant resolution. This can be seen by observing that following a barycentric subdivision at any of the junior points, the singularity on at least one of the resulting cones does not satisfy JunNec.

However, there does exist a crepant resolution; examples can be found using the Magma program described in Chapter 3.

The ideal situation would be to find a refinement of the conjecture to give necessary and sufficient conditions on the existence of a crepant resolution in dimension four and above. It is not clear how this could be achieved. Further understanding of the examples of Firla and Ziegler may help to find such a refinement.

2.6 Projective crepant resolutions

The quotient singularity $X/\left\langle \frac{1}{67}(1,5,8,53)\right\rangle$ cannot be resolved to a crepant resolution by a chain of blow-ups. We will see that this does not mean that it does not have a projective crepant resolution.

Let Σ be the toric fan of the resolution of $X/\left\langle \frac{1}{67}(1,5,8,53)\right\rangle$ consisting of the following cones

$e_4e_3e_2p_1$	$e_4e_3p_1p_{14}$	$e_4e_3p_{14}p_{27}$	$e_4e_3p_{27}e_1$	$e_4e_2p_1p_9$	$e_4e_2p_9p_{17}$
$e_4e_2p_{17}p_{42}$	$e_4e_2p_{42}e_1$	$e_4p_1p_9p_{17}$	$e_4p_1p_{14}p_{27}$	$e_4p_1p_{17}p_{42}$	$e_4p_1p_{27}e_1$
$e_4p_1p_{42}e_1$	$e_3e_2p_1p_2$	$e_3e_2p_2p_3$	$e_3e_2p_3p_4$	$e_3e_2p_4p_9$	$e_3e_2p_9p_{14}$
$e_3e_2p_{14}p_{19}$	$e_3e_2p_{19}p_{43}$	$e_3e_2p_{43}e_1$	$e_3p_1p_2p_{14}$	$e_3p_2p_3p_{14}$	$e_3p_3p_4p_{14}$
$e_3p_4p_9p_{14}$	$e_3p_{14}p_{19}p_{43}$	$e_3p_{14}p_{27}e_1$	$e_3p_{14}p_{43}e_1$	$e_2p_1p_2p_9$	$e_2p_2p_3p_9$
$e_2p_3p_4p_9$	$e_2p_9p_{14}p_{19}$	$e_2p_9p_{17}p_{42}$	$e_2p_9p_{19}p_{43}$	$e_2p_9p_{42}e_1$	$e_2p_9p_{43}e_1$
$p_1p_2p_9p_{17}$	$p_1p_2p_{14}p_{27}$	$p_1p_2p_{17}p_{27}$	$p_1p_{17}p_{27}p_{42}$	$p_1p_{27}p_{42}e_1$	$p_2p_3p_9p_{17}$
$p_2p_3p_{14}p_{27}$	$p_2p_3p_{17}p_{27}$	$p_3p_4p_9p_{18}$	$p_3p_4p_{14}p_{18}$	$p_3p_9p_{17}p_{18}$	$p_3p_{14}p_{18}p_{28}$
$p_3p_{14}p_{27}p_{28}$	$p_3p_{17}p_{18}p_{42}$	$p_3p_{17}p_{27}p_{42}$	$p_3p_{18}p_{27}e_1$	$p_3p_{18}p_{28}e_1$	$p_3p_{27}p_{28}e_1$
$p_4p_9p_{14}p_{18}$	$p_9p_{14}p_{18}p_{28}$	$p_9p_{14}p_{19}p_{28}$	$p_9p_{17}p_{18}p_{42}$	$p_9p_{18}p_{28}p_{43}$	$p_9p_{18}p_{42}e_1$
$p_9p_{18}p_{43}e_1$	$p_9p_{19}p_{28}p_{43}$	$p_{14}p_{19}p_{28}p_{43}$	$p_{14}p_{27}p_{28}e_1$	$p_{14}p_{28}p_{43}e_1$	$p_{18}p_{27}p_{42}e_1$
$p_{18}p_{28}p_{43}e_1.$					

We will show that this resolution is projective by showing that the ample cone is nonempty.

We have the short exact sequence

$$0 \to M \xrightarrow{A} \mathbb{Z}^{\Sigma(1)} \xrightarrow{B} \mathrm{Pic}(X_\Sigma) \to 0$$

where $\Sigma(1)$ is the set of rays of the fan Σ,

$$A = \begin{pmatrix} 1 & 5 & 8 & 53 \\ 2 & 10 & 16 & 39 \\ 3 & 15 & 24 & 25 \\ 4 & 20 & 32 & 11 \\ 9 & 45 & 5 & 8 \\ 14 & 3 & 45 & 5 \\ 17 & 18 & 2 & 30 \\ 18 & 23 & 10 & 16 \\ 19 & 28 & 18 & 2 \\ 27 & 1 & 15 & 24 \\ 28 & 6 & 23 & 10 \\ 42 & 9 & 1 & 15 \\ 43 & 14 & 9 & 1 \\ 67 & 0 & 0 & 0 \\ 0 & 67 & 0 & 0 \\ 0 & 0 & 67 & 0 \\ 0 & 0 & 0 & 67 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & -8 & 0 & 3 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -6 & 0 & 9 & 1 & -4 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & -12 & 0 & 18 & 0 & -7 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & -16 & 0 & 24 & 0 & -9 & -5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 & 0 & 0 & 0 & -30 & 0 & 45 & 0 & -17 & -9 & 5 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & -60 & 1 & 90 & 0 & -34 & -18 & 11 & 0 \\ 0 & 0 & 0 & 1 & 11 & 0 & 0 & 0 & 0 & -66 & 0 & 99 & 0 & -37 & -20 & 12 & 0 \\ 0 & 0 & 0 & 0 & 16 & 0 & 0 & 1 & 0 & -96 & 0 & 144 & 0 & -54 & -29 & 18 & 0 \\ 0 & 0 & 1 & 0 & 25 & 0 & 0 & 0 & 0 & -150 & 0 & 225 & 0 & -84 & -45 & 28 & 0 \\ 0 & 0 & 0 & 0 & 30 & 0 & 1 & 0 & 0 & -180 & 0 & 270 & 0 & -101 & -54 & 34 & 0 \\ 0 & 1 & 0 & 0 & 39 & 0 & 0 & 0 & 0 & -234 & 0 & 351 & 0 & -131 & -70 & 44 & 0 \\ 1 & 0 & 0 & 0 & 53 & 0 & 0 & 0 & 0 & -318 & 0 & 477 & 0 & -178 & -95 & 60 & 0 \\ 0 & 0 & 0 & 0 & 67 & 0 & 0 & 0 & 0 & -402 & 0 & 603 & 0 & -225 & -120 & 76 & 1 \end{pmatrix}$$

is found by calculating the Smith normal form of A.

We want to calculate the nef cone of X_Σ. In toric geometry this is just the cone of globally generated divisors. Combinatorially this corresponds to

$$\bigcap_{\sigma\in\Sigma} \mathrm{pos}([D_i] : i \notin \sigma).$$

The columns of B correspond to the divisors D_i, so for each cone of the resolution we take B_σ to be the submatrix containing the columns B_i of B such that i is not in σ.

Consider a cone σ in N. Its rays are generated by 4 rows of A; label these $\alpha, \beta, \gamma, \delta$. Now take $B_\sigma = (b_{i,j})$ where $1 \leq i \leq 13, j \neq \alpha, \beta, \gamma, \delta$, that is, all columns of B except columns $\alpha, \beta, \gamma, \delta$.

We invert B_σ to find the equations of the hyperplanes defining the 13 dimensional nef cone in $\mathrm{Pic}(X_\Sigma)$.

Let σ be the cone $p_1p_2p_9p_{17}$, generated by the rays $\frac{1}{67}(1,5,8,53)$, $\frac{1}{67}(2,10,16,39)$, $\frac{1}{67}(9,45,5,8)$, $\frac{1}{67}(17,18,2,30)$. These correspond to the first, second, fifth and seventh rows of A. We take B_σ to be B with the first, second, fifth and seventh columns omitted

$$B_\sigma = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 5 & 0 & -8 & 0 & 3 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -6 & 0 & 9 & 1 & -4 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & -12 & 0 & 18 & 0 & -7 & -4 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & -16 & 0 & 24 & 0 & -9 & -5 & 3 & 0 \\
0 & 0 & 1 & 0 & 0 & -30 & 0 & 45 & 0 & -17 & -9 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & -60 & 1 & 90 & 0 & -34 & -18 & 11 & 0 \\
0 & 1 & 0 & 0 & 0 & -66 & 0 & 99 & 0 & -37 & -20 & 12 & 0 \\
0 & 0 & 0 & 1 & 0 & -96 & 0 & 144 & 0 & -54 & -29 & 18 & 0 \\
1 & 0 & 0 & 0 & 0 & -150 & 0 & 225 & 0 & -84 & -45 & 28 & 0 \\
0 & 0 & 0 & 0 & 0 & -180 & 0 & 270 & 0 & -101 & -54 & 34 & 0 \\
0 & 0 & 0 & 0 & 0 & -234 & 0 & 351 & 0 & -131 & -70 & 44 & 0 \\
0 & 0 & 0 & 0 & 0 & -318 & 0 & 477 & 0 & -178 & -95 & 60 & 0 \\
0 & 0 & 0 & 0 & 0 & -402 & 0 & 603 & 0 & -225 & -120 & 76 & 1
\end{pmatrix}$$

We use the inverse,

$$B_\sigma^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -3 & 2 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & -1 & -5 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 2 & 0 \\ -3 & 0 & 0 & 11 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & -2 & -3 & 3 & 0 \\ -2 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & -3 & -1 & 2 & 0 \\ 0 & 1 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & -3 & -2 & 3 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & -5 & -2 & 4 & 0 \\ 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & -7 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{pmatrix}$$

of B_σ to find the hyperplanes defining the cone:

$$\begin{array}{ll} x_9 - 2x_{11} + x_{12} \geq 0, & x_7 - 3x_{11} + 2x_{12} \geq 0, \\ 3x_4 + x_5 - x_{10} - 5x_{11} + 4x_{12} \geq 0, & x_8 - x_{10} - x_{11} + x_{12} \geq 0, \\ x_3 - x_{10} - 2x_{11} + 2x_{12} \geq 0, & -3x_1 + 11x_4 - 2x_{10} - 2x_{11} + 2x_{12} \geq 0, \\ 3x_4 + x_6 - 2x_{10} - 3x_{11} + 3x_{12} \geq 0, & -2x_1 + 8x_4 - 3x_{10} - x_{11} + 2x_{12} \geq 0, \\ x_2 + 3x_4 - 3x_{10} - 2x_{11} + 3x_{12} \geq 0, & 6x_4 - 5x_{10} - 2x_{11} + 4x_{12} \geq 0, \\ -6x_4 + x_{10} + x_{11} - x_{12} \geq 0, & 3x_4 - 7x_{11} + 5x_{12} \geq 0, \\ x_{11} - 2x_{12} + x_{13} \geq 0. & \end{array}$$

We take the union of all the hyperplanes from each σ in Σ, which gives us the cone of globally generated divisors. PORTA [CL09] converts the hyperplanes into the rays of the cone.

This cone is 13 dimensional, so we choose 13 rays which generate the cone:

$$\begin{aligned}
&(-3,0,1,2,3,7,9,13,22,32,41,60,79),\\
&(-3,2,4,6,11,23,25,37,58,72,93,128,163),\\
&(0,-6,-10,-12,-24,-46,-48,-70,-106,-120,-155,-204,-252),\\
&(0,-6,-10,-12,-24,-46,-48,-70,-106,-119,-155,-204,-252),\\
&(0,-3,-5,-6,-12,-23,-24,-35,-53,-60,-78,-103,-128),\\
&(0,-3,-5,-6,-12,-23,-24,-35,-53,-60,-78,-103,-127),\\
&(0,-3,-5,-6,-12,-23,-24,-35,-53,-60,-78,-102,-126),\\
&(0,-2,-4,-5,-10,-19,-20,-30,-45,-50,-65,-85,-105),\\
&(0,-1,-2,-2,-4,-8,-8,-12,-18,-20,-26,-34,-42),\\
&(0,-1,-1,-1,-2,-4,-4,-6,-9,-10,-13,-17,-21),\\
&(1,-4,-8,-10,-19,-37,-40,-59,-90,-104,-135,-180,-225),\\
&(2,-6,-11,-14,-27,-53,-57,-83,-128,-148,-192,-256,-320),\\
&(2,-3,-6,-8,-15,-30,-33,-48,-74,-88,-114,-154,-194),
\end{aligned}$$

We take their sum to find a point v in the interior of the cone:

$$v=(-1,-36,-62,-74,-147,-282,-296,-433,-655,-735,-955,-1254,-1550),$$

and pull this back to a point w in $\mathbb{Z}^{\Sigma(1)}$:

$$w=(140,6,-125,-79,-79,-73,3,-122,-40,0,-110,0,-35,0,-3,0,283).$$

Since w is in the interior of the cone

$$\begin{aligned}
D=140D_1+6D_2-125D_3-79D_4-79D_5-73D_6+3D_7-122D_8-40D_9\\
+0D_{10}-110D_{11}+0D_{12}-35D_{13}+0D_{14}-3D_{15}+0D_{16}+283D_{17}
\end{aligned}$$

is certainly an ample divisor and so the resolution is projective.

2.7 Other subdivisions

An obvious alternative to the question "when does a crepant resolution exist?" is the question "why do some singularities have no crepant resolution?". Terminal singularities, $\frac{1}{r}(i,r-i,j,r-j)$, do not have a crepant resolution since there are no non-trivial junior points in the lattice $L=\mathbb{Z}^4+\frac{1}{r}(i,r-i,j,r-j)\cdot\mathbb{Z}$. Instead we consider the octahedron containing all the age 2 points, that is the octahedron

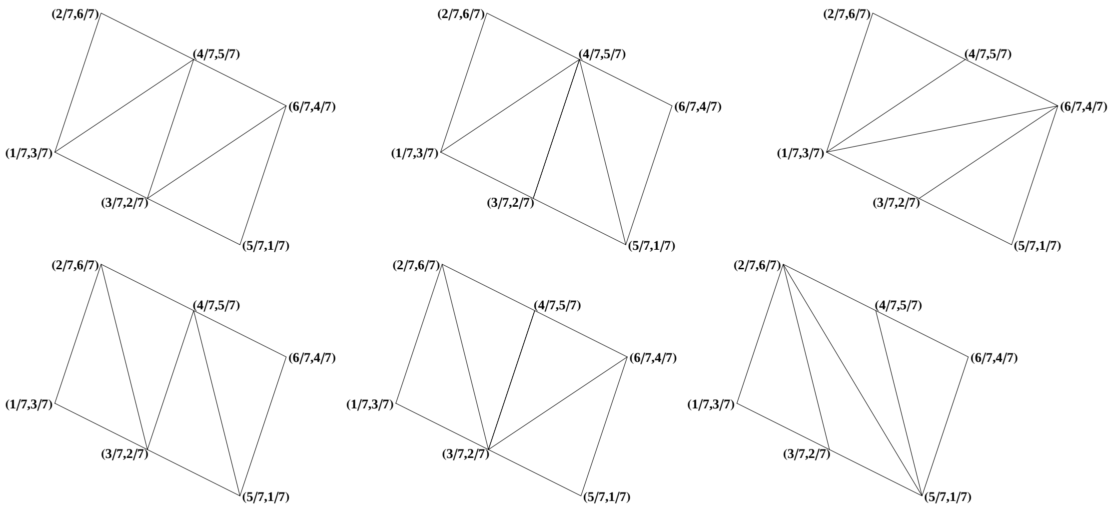

Figure 2.1: Triangulations of the rectangle q_1, q_2, q_5, q_6

with vertices $e_i + e_j$ for $i, j = 1, \dots 4,\ i \neq j$.

However a subdivision which includes any of the vertices of the simplex can not correspond to an economic resolution [Rei87].

Example 2.7.1. Let $L = \mathbb{Z}^4 + \frac{1}{7}(1,6,3,4) \cdot \mathbb{Z}$. There are six points inside the age 2 octahedron:

$$q_1 = \frac{1}{7}(1,6,3,4), \qquad q_2 = \frac{1}{7}(2,5,6,1), \qquad q_3 = \frac{1}{7}(3,4,2,5),$$
$$q_4 = \frac{1}{7}(4,3,5,2), \qquad q_5 = \frac{1}{7}(5,2,1,6), \qquad q_6 = \frac{1}{7}(1,6,4,3).$$

These points are clearly coplanar and q_1, q_2, q_5, q_6 are the vertices of a rectangle. The six ways to triangulate this rectangle are shown in Figure 2.1, with the rectangle being drawn in the x_1-x_3 plane.

Claim 2.7.2. *There is no way to tessellate the first orthant using only basic cones.*

We calculate all the basic simplices. There are only four simplices with one interior point:

$$\Delta_1 = q_1 e_2 e_3 e_4,\ \Delta_2 = q_2 e_1 e_2 e_3,\ \Delta_3 = q_5 e_1 e_2 e_4,\ \Delta_4 = q_6 e_1 e_3 e_4.$$

Start by assuming that we must have all of these. By comparing faces of the Δ_i with faces of the other basic simplices we can uniquely find neighbours for some of the faces. At some point we must choose a triangulation on the rectangle

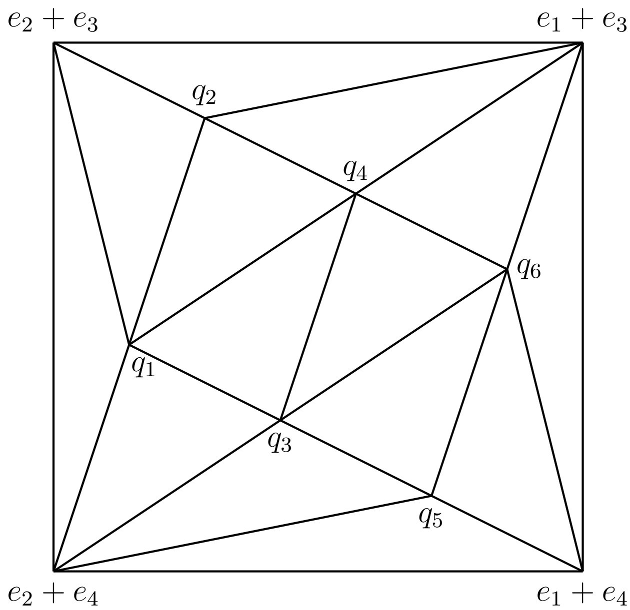

Figure 2.2: Triangulation of the x_1x_3-plane in the age 2 octahedron

q_1, q_2, q_5, q_6. We will use

$$q_1q_2q_4, \quad q_1q_3q_4, \quad q_3q_4q_6, \quad q_3q_5q_6.$$

The simplex Δ_1 contains the face $q_1e_3e_4$. We take $\Delta_5 = q_1q_2e_3e_4$ since it is the only other simplex to contain this face. Now Δ_5 contains the face $q_2e_3e_4$, so must have $\Delta_6 = q_2q_3e_3e_4$ as its neighbour. The simplex Δ_6 contains the face $q_2q_3e_4$, but because of our choice of triangulation, there is no other simplex containing this face. So it is not possible to cover the space using this triangulation of the rectangle $q_1q_2q_5q_6$. Further analogous calculations show that none of the triangulations lead to a triangulation of the whole space. Bouvier and Gonzalez-Sprinberg [BGS95] also show that this example does not have a crepant resolution.

It is possible to find a triangulation using $\frac{1}{2}(1,1,1,1)$ simplices. We start by choosing a triangulation of the x_1-x_3 plane in the age 2 octahedron, as shown Figure 2.2. We turn each of these triangles into two tetrahedra by adding the vertex $e_1 + e_2$ or the vertex $e_3 + e_4$ respectively.

Comparing faces and checking the dimensions and singularity of possible tetrahedra shows that we need the four tetrahedra $f_{12}f_{23}f_{13}e_2$, $f_{12}f_{13}f_{14}e_1$, $f_{34}f_{23}f_{13}e_3$, $f_{34}f_{24}f_{14}e_4$ to tile the first orthant. Here $f_{ij} := e_i + e_j$.

There is a question as to whether or not crepant resolution is the correct thing to do in dimension four and above, but this does not seem to help answer that question.

In a further attempt to answer the question "why do some singularities have no crepant resolution?". We move to an example which is not terminal, but still appears to have no crepant resolution due to having too few junior points. We explored ways of finding additional points which would allow a subdivision into basic cones.

Suppose, instead of taking barycentric subdivision at a point, we do a similar operation around a line segment. Consider the line segment p_1p_2 in the tetrahedral simplex $e_1e_2e_3e_4$. Let H_1 be the plane containing p_1, p_2, e_1 and H_2 be the plane containing p_1, p_2, e_2. Consider the pencil of planes $\lambda H_1 + \mu H_2$. If p_1, p_2 are not collinear with any of the e_i and the line segment p_ip_j is not parallel to an edge e_ie_j then, as μ and λ vary the pencil of planes sweeps through each of the vertices e_i in turn. Thus we take the cones $p_1p_2e_ie_j$ where e_i and e_j are consecutive as $\lambda H_1 + \mu H_2$ sweeps through. This gives a partial subdivision of the original tetrahedron.

The question we wanted to answer was: in the case where there are only two junior points p_1, p_2, does extending the line segment p_1p_2 to the faces of the tetrahedron provide a resolution into basic simplices. This would allow us to consider chains of such subdivisions. However, the answer to the question is no, as the following examples will illustrate.

Example 2.7.3. Consider the quotient singularity $\frac{1}{17}(1,3,5,8)$, which does not have a crepant resolution. Let $L = \mathbb{Z}^4 + \frac{1}{17}(1,3,5,8) \cdot \mathbb{Z}$. This has two junior points in the interior of the junior simplex:

$$p_1 = \tfrac{1}{17}(1,3,5,8), \quad p_7 = \tfrac{1}{17}(7,4,1,5).$$

Take the four hyperplanes containing p_1, p_7 and one of the e_i:

$$\begin{array}{ll} H_1 : (x_2 + x_3 - x_4 = 0) & H_2 : (x_1 + 3x_3 - 2x_4 = 0) \\ H_3 : (x_1 - 3x_2 + x_4 = 0) & H_4 : (x_1 - 2x_2 + x_3 = 0) \end{array}$$

The pencil of planes $\lambda H_1 + \mu H_2$ sweeps through each of the vertices e_i in turn. By projecting to a plane perpendicular to the line we see that the order the pencil sweeps through the vertices is e_1, e_3, e_2, e_4. This tells us that subdividing around the line segment p_1p_7 gives the cones:

Cone	Singularity
$e_1p_1p_7e_4$	–
$e_1p_1e_3p_7$	–
$p_1e_2e_3p_7$	$\frac{1}{3}(1,2,1,2)$
$p_1e_2p_7e_4$	$\frac{1}{2}(1,1,1,1)$

These do not cover the whole of the junior simplex. We add in the cones at either end of the line segment $p_1e_2e_3e_4$ and $e_1e_2p_7e_4$, and compare faces to find any missing cones. We now have:

Cone	Singularity
$p_1e_2e_3e_4$	$-$
$e_1e_2p_7e_4$	$-$
$e_1e_2e_3p_7$	$\frac{1}{5}(1,4,2,3)$
$e_1p_1e_3e_4$	$\frac{1}{3}(1,2,1,2)$

The line segment can be extended to meet the face $e_2e_3e_4$ and $e_1e_2e_4$ at the points $\frac{1}{17}(0,\frac{17}{6},\frac{34}{6},\frac{51}{6})$ and $\frac{1}{17}(\frac{17}{2},\frac{17}{4},0,\frac{17}{4})$ respectively. This does not help us to subdivide the cones further as these points are contained within basic cones.

The only way to subdivide $p_1e_2p_7e_4$ into two basic cones would be to subdivide at a point on one of the edges $p_1e_2, p_1p_7, p_1e_4, e_2p_7, e_2e_4$, or p_7e_4. Each of these edges, however, are common to at least one basic cone, and adding an additional point would also require further subdivision of the basic cone. It is not clear how to proceed from here.

It may be worth noting that the cones with $\frac{1}{3}(1,2,1,2)$ singularity can be further subdivided by taking a point in the faces $e_2e_3p_7$ and $e_1e_3e_4$. These faces are not common to any other cone.

We now look at an example where the points p_1 and p_2 are collinear with one of the e_i.

Example 2.7.4. Let $L = \mathbb{Z}^4 + \frac{1}{8}(1,1,1,5)\cdot\mathbb{Z}$. The two junior points in the interior of the junior simplex

$$p_1 = \tfrac{1}{8}(1,1,1,5), \quad p_2 = \tfrac{1}{8}(2,2,2,2).$$

are collinear with the vertex e_4.

Subdivision around the line segment p_1p_2 gives:

Cone	Singularity
$e_1p_1e_3p_2$	$-$
$p_1e_2e_3p_2$	$-$
$e_1e_2p_1p_2$	$-$
$p_1e_2e_3e_4$	$-$
$e_1p_1e_3e_4$	$-$
$e_1e_2p_1e_4$	$-$
$e_1e_2e_3p_2$	$\frac{1}{2}(1,1,1,1)$

The idea was that we could subdivide the cone $e_1e_2e_3p_2$ at the point where the extended line segment p_1, p_2 meets the face $e_1e_2e_3$. This is the point $\frac{1}{8}(\frac{8}{3}, \frac{8}{3}, \frac{8}{3}, 0)$, which is contained in $e_1e_2e_3p_2$. However subdivision at this point cannot give two basic cones.

Subdivision at the point $\frac{1}{8}(0, 4, 4, 0)$ (respectively $\frac{1}{8}(4, 0, 4, 0)$, $\frac{1}{8}(4, 4, 0, 0)$) would give a resolution into eight basic cones, which is what we wanted. However, again, any of these would require further subdivision of a basic cone.

Chapter 3

Resolution algorithm

In order to find all crepant resolutions of a particular singularity I have written a MAGMA program to find triangulations of the junior simplex. The code is written from first principles and relies on only basic MAGMA functions.

This chapter is organised as follows. I start by giving an idea of what the program does, followed by a more detailed description of the program which is illustrated by a flowchart. I use pseudocode to discuss the details of each algorithm in turn. The MAGMA program is available at `http://www.warwick.ac.uk/staff/S.E.Davis/Thesis/ResolutionAlgorithm.m`. Examples of the program in action are given at the end of the chapter.

Throughout this section we will use typewriter-style to distinguish variables in MAGMA, for example `CrepantCones`, and small caps to distinguish functions, for example HASCREPRES.

3.1 Overview of the program

3.1.1 Short description of the program

We find all the basic cones having vertices in the set of junior points of the singularity (this includes $e_1 = (1, 0, 0, 0)$ etc.). We then find cones whose faces form part of the faces of the junior simplex, and choose a subset of these which cover all the faces of the junior simplex. We continue by finding the neighbours of each cone that has already been chosen. We use a decision tree to choose between these neighbours until we obtain a crepant resolution or prove the nonexistence of one.

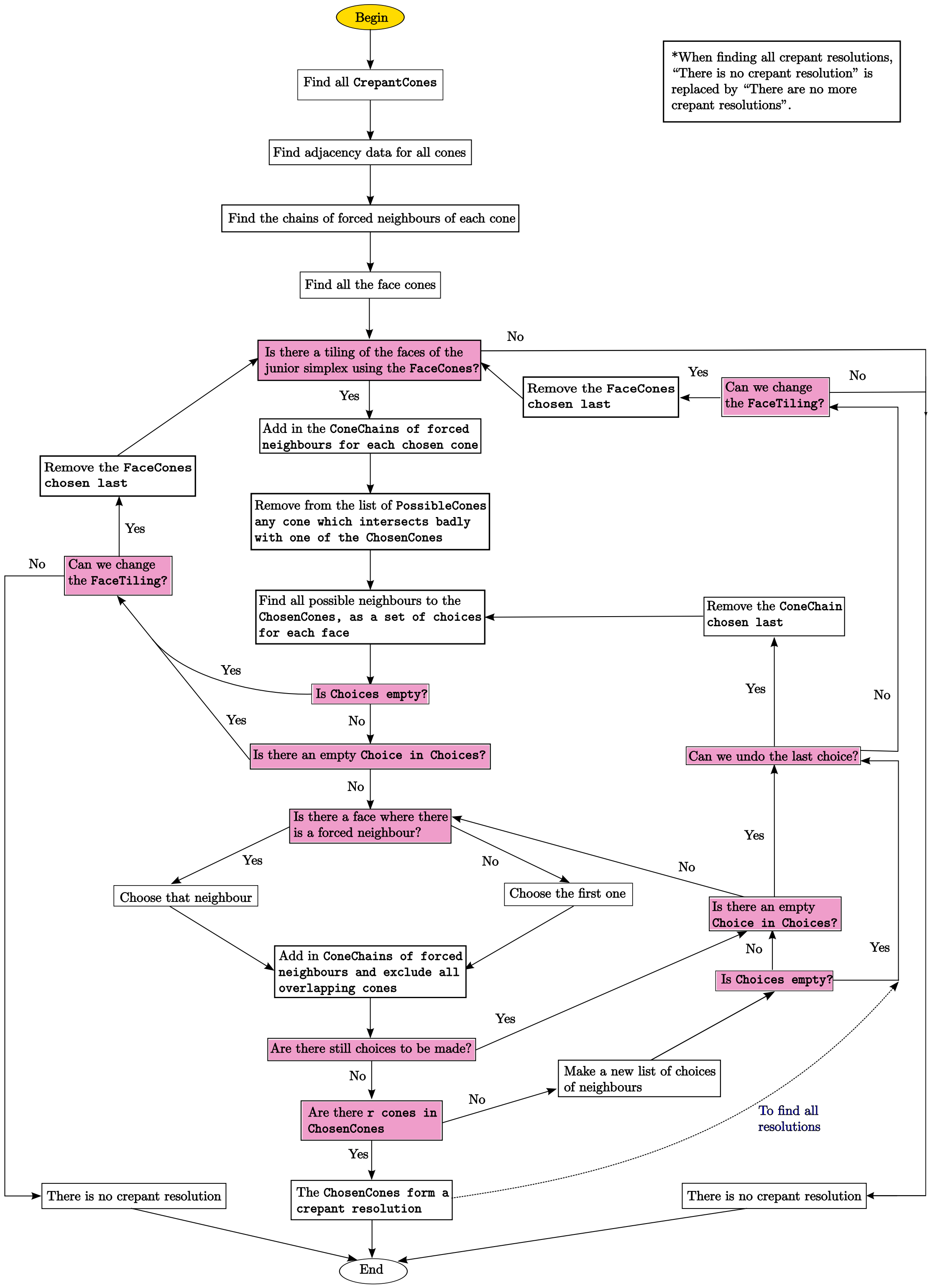

Figure 3.1: Flowchart for the resolution algorithm

3.1.2 Detailed description of the program

Let G be the cyclic subgroup of $\mathrm{SL}(4,\mathbb{C})$ generated by $\frac{1}{r}(a_1,a_2,a_3,a_4)$, and let X be the quotient singularity $\mathbb{C}^4/G$.

Definition 3.1.1. [IR96] Let $L = \mathbb{Z}^4 + \frac{1}{r}(a_1,a_2,a_3,a_4)\cdot\mathbb{Z}$ be a lattice. Define the *age* of a point (b_1,b_2,b_3,b_4) of L to be

$$\sum_{i=1}^{4} b_i.$$

Since all of our groups are in SL the age of every lattice point will be an integer. We call the points with age 1 *junior points*.

Denote by $\overline{ka_1}$ the integer $ka_1 \mod r$. The junior points of L are the points $\frac{1}{r}(\overline{ka_1},\overline{ka_2},\overline{ka_3},\overline{ka_4})$, for $1 \le k < r$, such that $\frac{1}{r}\sum_{i=1}^{4}\overline{ka_i} = 1$, together with the points

$$e_1 = (1,0,0,0), \quad e_2 = (0,1,0,0), \quad e_3 = (0,0,1,0), \quad e_4 = (0,0,0,1).$$

If $a_1 = 1$ we refer to the junior point $\frac{1}{r}(a,b,c,d)$ as p_a, where a is the first coordinate of the point.

These points all lie on the plane $x_1+x_2+x_3+x_4 = 1$. We refer to the intersection of this plane with the first orthant as the *junior simplex*. This is a tetrahedron whose vertices are the points e_1,e_2,e_3,e_4.

Recall from Remark 1.2.8 that the existence of a crepant resolution $f\colon Y \to X$ is equivalent to the 1-skeleton of the toric fan of Y consisting only of junior points.

The following description of how the program works is illustrated by Figure 3.1, which shows the main steps in the algorithm. Further details are given in sections 3.2–3.4.

We find all cones of the form $p_1p_2p_3p_4$, where p_i are any junior points. Such a cone is basic if the vectors p_1,p_2,p_3,p_4 generate the lattice L. We calculate the determinant of the matrix

$$\begin{pmatrix} p_{1,1} & p_{1,2} & p_{1,3} & p_{1,4} \\ p_{2,1} & p_{2,2} & p_{2,3} & p_{2,4} \\ p_{3,1} & p_{3,2} & p_{3,3} & p_{3,4} \\ p_{4,1} & p_{4,2} & p_{4,3} & p_{4,4} \end{pmatrix} \tag{3.1}$$

where $p_{1,i}$ denotes the ith coordinate of p_1. This is the volume of the parallelepiped with vertices p_1, p_2, p_3, p_4, whose volume is 4! times that of the simplex $p_1p_2p_3p_4$. The condition that the vectors p_1, p_2, p_3, p_4 generate the lattice L is equivalent to $\frac{1}{r}$ times the determinant of matrix (3.1) being equal to ± 1. If the cone $p_1p_2p_3p_4$ is basic, we say the simplex $p_1p_2p_3p_4$ has relative volume 1. The junior simplex has relative volume r so a resolution requires r cones. In Magma we store the set of all basic cones in an ordered list called `CrepantCones`.

Two crepant cones are *neighbours* if they share a common face and have opposite orientation. That is, they have three vertices in common, but the signs of the determinants of their matrices as in (3.1) are opposite. A search through the faces of every cone in `CrepantCones` reveals all pairs of cones with a shared common face. A further check on the determinants of these cones shows which of these pairs are neighbours. We will store this information in a table called the `AdjacencyGraph`. We fix an order on the vertices of each cone and number the faces by the index of the missing vertex; for example, the face $p_2p_3p_4$ is considered to be face 1 of the cone $p_1p_2p_3p_4$. Let i, j run through `CrepantCones`. In the (i, j)th entry of the `AdjacencyGraph` we record the index of the common face of cones i and j, as labelled in cone i, if they are neighbours, and we record a 0 if the cones i and j are not neighbours (this includes the case $i = j$). Note that the table is not symmetric.

Example 3.1.2. Let $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$, $e_4 = (0, 0, 0, 1)$, $p_1 = \frac{1}{7}(1, 1, 1, 4)$ and $p_2 = \frac{1}{7}(2, 2, 2, 1)$. Consider the cones

$$C_1 = e_1e_2p_1p_2, \quad C_2 = e_1p_1e_3p_2, \quad C_3 = p_1e_2e_3p_2, \quad C_4 = e_1e_2p_1e_4,$$
$$C_5 = e_1p_1e_3e_4, \quad C_6 = p_1e_2e_3e_4, \quad C_7 = e_1e_2e_3p_2.$$

Cones C_1, C_2 share a common face $e_1p_1p_2$. Since e_2 is the vertex missing from C_1, this face is considered to be the second face of C_1 (it is the third face of C_2). The first face of C_1, that is $e_2p_1p_2$, is common to C_3, the third face is common to C_7 and the fourth face is common to the C_4. It is not hard to check that these pairs of cones are actually neighbours. The first cone has no face in common with the fifth or sixth cone. This information is recorded in the first row of the `AdjacencyGraph`, Table 3.1. The $(1, 2)$th entry tells us that C_1 shares its second face with C_2. The 3 in the $(2, 1)$th entry shows that the common face of cones C_2 and C_1 is the third face of C_2. In this example the `AdjacencyGraph` tells us a unique triangulation of the junior simplex as every pair of neighbours is unique. Note that cones C_4, C_5, C_6, C_7 have only three neighbours because one of each of

```
[     [ 0, 2, 1, 4, 0, 0, 3 ],
      [ 3, 0, 1, 0, 4, 0, 2 ],
      [ 3, 2, 0, 0, 0, 4, 1 ],
      [ 4, 0, 0, 0, 2, 1, 0 ],
      [ 0, 4, 0, 3, 0, 1, 0 ],
      [ 0, 0, 4, 3, 2, 0, 0 ],
      [ 3, 2, 1, 0, 0, 0, 0 ]     ]
```

Table 3.1: `AdjacencyGraph` for Example 3.1.2

their faces is a face of the junior simplex.

The algorithm uses the `AdjacencyGraph` to find the number of cones at each face of a given cone. If cone C_1 has a face with exactly one neighbouring cone C_2, we know that if C_1 is part of a resolution then C_2 must also be part of that resolution. Note that it is possible that C_1 is not the only neighbour of C_2 at their common face, so the converse is not true. If C_2 is the only neighbour to C_1 at a given face we say that C_2 is a *forced neighbour of* C_1. By considering forced neighbours and the forced neighbours of forced neighbours of a cone C, we can find a set of cones which must belong to any resolution containing C. We store the set of all such cones in a list called the `ConeChain` of C.

For every basic cone C there exists a `ConeChain` (of length at least one). In Example 3.1.2, the `ConeChain` of the first cone contains all seven cones. Cones 2, 3, 4 and 7 are the unique neighbours at faces 2, 1, 4 and 3 of the first cone respectively. Cone 5 is a unique neighbour at the fourth face of cone 2 and cone 6 is a unique neighbour at the fourth face of cone 3.

We are looking for a triangulation of the junior simplex. So far we know how the cones fit together, but we must also make sure we choose cones which do not overlap. We use the toric geometry definitions in Magma to check the dimension of the intersection of every pair of crepant cones. If the dimension is 4 then the cones have nontrivial intersection and cannot appear in the same resolution. We record this information in a table called the `OverlapGraph`.

We use the `OverlapGraph` to make a table called `ConeChainOverlapGraph` which tells us whether or not two cone chains have nontrivial intersection. We check whether the `ConeChain` of cone C contains cones which overlap — i.e. does the `ConeChain` go over itself? If the `ConeChain` of C overlaps itself it cannot be part of the resolution. We make a list called `AllowedCones` containing all cones whose `ConeChain` do not overlap with themselves.

We find the set of junior points which lie on the four triangular faces of the junior simplex. We find all the crepant cones which have three vertices from this set. These cones have a face at a face of the junior simplex, and as such, have at most three neighbours. We call the set of all such cones `FaceCones`.

We choose the first `FaceCone`, C, and create a list `FaceTiling`, to which we add the cone C. We ignore any `FaceCones` which have a non-proper intersection with any of the `ConeChains` of the cones in `FaceTiling`. We continue choosing cones from `FaceCones` until all have been chosen or discarded.

If we don't find a `FaceTiling`, a different cone is chosen first until all options have been tried, if there is still no `FaceTiling` then no crepant resolution exists. Otherwise, let `ChosenCones` be the list containing all the cones in `FaceTiling` and in the `ConeChains` of these cones.

We exclude from `AllowedCones` any cones which have a non-proper intersection with a cone in `ChosenCones`.

If `ChosenCones` contains r cones we are done, otherwise we need to make a choice of neighbour at each of the remaining faces of the `ChosenCones`.

We find the subset of `AllowedCones` which have a common face with at least one of the `ChosenCones`. For each face of the cones in `ChosenCones`, we find the cones which also contain this face. If none of these cones are already in the `ChosenCones` we need to choose between them. (We also check whether the cones are valid — i.e. do they not overlap with anything in `ChosenCones` — at this stage. This will be described in more detail later.) We save each of these sets in a list called `Choices`.

If any of the sets in `Choices` is empty or `Choices` itself is empty then we must choose a different face tiling. If `Choices` is nonempty and if any of the sets in `Choices` contains exactly one cone, C, then C is forced and we add the `ConeChain` of C to `ChosenCones`.

We remove from `Choices` every set containing any element of the `ConeChain` of C and remove from every set in `Choices` any cone that was overlapping with a newly chosen one. Then we remove from the sets of `Choices` any cone which overlaps with the `ConeChain` of C.

If every set of `Choices` contains at least two cones, we choose the first cone from the first set, and follow the same procedure as for the forced cone C above. We continue to choose cones until `Choices` is empty or we have an empty set in `Choices`.

If `Choices` is empty and we do not have r cones, we must make some more sets of choices. The algorithm for doing this will be explained in section 3.4. If the new `Choices` is nonempty we can continue to make choices as before. If `Choices` is empty or contains an empty set we undo the last choice we made. We remove the last cone chosen from `ChosenChones` and make a new list `Choices`. We continue as before.

If we have undone all the choices we made, we can try to find a different `FaceTiling`. If this is possible we run through the algorithm again. Otherwise, we have been unable to find a resolution.

The algorithm will also find all crepant resolutions. In this case, once a resolution has been found it undoes the last choice made, and continues as above.

Example 3.1.3. Consider the singularity $\frac{1}{17}(1,1,6,9)$. There are five junior points in the lattice $L = \mathbb{Z}^4 + \frac{1}{17}(1,1,6,9)\cdot\mathbb{Z}$:

$$p_1 := \frac{1}{17}(1,1,6,9), \quad p_2 := \frac{1}{17}(2,2,12,1), \quad p_3 := \frac{1}{17}(3,3,1,10),$$
$$p_4 := \frac{1}{17}(4,4,7,2), \quad p_6 := \frac{1}{17}(6,6,2,3).$$

We find 33 basic cones

$$\begin{array}{llllll}
p_1e_2e_3e_4 & p_2p_1e_2e_3 & p_3p_1e_2e_4 & p_3p_1e_2e_3 & p_3p_2e_2e_3 & p_3p_2p_1e_2 \\
p_4p_1e_2e_4 & p_4p_2p_1e_2 & p_4p_3e_2e_4 & p_4p_3p_1e_2 & p_4p_3p_2e_2 & p_6p_3p_1e_2 \\
p_6p_4p_1e_2 & p_6p_4p_3e_2 & e_1p_1e_3e_4 & p_2e_1e_2e_3 & e_1p_2p_1e_3 & e_1p_3e_2e_4 \\
e_1p_3p_1e_4 & e_1p_3p_1e_3 & e_1p_3p_2e_3 & e_1p_3p_2p_1 & e_1p_4p_1e_4 & e_1p_4p_2e_2 \\
e_1p_4p_2p_1 & e_1p_4p_3e_4 & e_1p_4p_3p_1 & e_1p_4p_3p_2 & e_1p_6p_3e_2 & e_1p_6p_3p_1 \\
e_1p_6p_4e_2 & e_1p_6p_4p_1 & e_1p_6p_4p_3. & & &
\end{array}$$

By considering the faces of these we find the four cones which have a face on the faces of the junior simplex:

$$p_1e_2e_3e_4, \quad e_1p_1e_3e_4, \quad e_1p_3e_2e_4, \quad p_2e_1e_2e_3.$$

We want to find the neighbours of each of these cones. We look to see which cones share a common face with the fourth cone, $p_2e_1e_2e_3$. We set an order on the vertices and number the faces by the index of the vertex missing from $p_2e_1e_2e_3$, and record the number of the common face in the `AdjGraph` :

```
[ 0, 0, 0, 0, 3, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 3, 2, 4, 0, 0, 0, 0, 0, 0, 0, 0 ]
```

This tells us that there is a choice of neighbour at the second and third faces, but a unique neighbour at the fourth face. This unique neighbour turns out to be the cone $p_2e_1e_2p_4$. We find that the `ConeChain` of $p_2e_1e_2e_3$ is

$$\{p_2e_1e_2e_3, \quad e_1e_2p_4p_2, \quad e_1e_2p_4p_6, \quad e_1e_2p_3p_6, \quad e_1p_3e_2e_4\}.$$

The union of the `ConeChains` of the edge pieces give us the `ChosenCones`:

$$p_2e_1e_2e_3, \quad e_1e_2p_4p_2, \quad e_1e_2p_4p_6, \quad e_1e_2p_3p_6, \quad e_1p_3e_2e_4, \quad p_1e_2e_3e_4, \quad e_1p_1e_3e_4. \tag{3.2}$$

We must now make choices as to which neighbour we should take at each face. For $p_2e_1e_2e_3$ we have a choice between the cones $e_1p_1e_3p_2$ and $e_1p_3e_3p_2$. Choosing $e_1p_1e_3p_2$, we get the `ConeChain`

$$\begin{gathered} p_1e_2e_3e_4, \quad p_2p_1e_2e_3, \quad e_1p_1e_3e_4, \quad e_1p_2e_2e_3, \quad e_1p_2p_1e_3, \\ e_1p_3e_2e_4, \quad e_1p_4p_2e_2, \quad e_1p_6p_3e_2, \quad e_1p_6p_4e_2, \end{gathered}$$

which forces us to take

$$e_1p_1e_3p_2, \quad p_1e_2e_3p_2 \tag{3.3}$$

as all the other cones in the `ConeChain` are already in the `ChosenCones`. We choose $p_1e_2p_4p_6$ over $p_3e_2p_4p_6$ as a neighbour for $e_1e_2p_4p_6$. This has `ConeChain`

$$\begin{gathered} p_1e_2p_4p_6, \quad p_1e_2p_4p_2, \quad e_1p_1p_4p_2, \quad e_1p_1p_4p_6, \\ e_1p_1p_3p_6, \quad e_1p_1p_3e_4, \quad p_1e_2p_3e_4, \quad p_1e_2p_3p_6, \end{gathered} \tag{3.4}$$

Thus there are seventeen cones: the seven cones of (3.2), the two cones of (3.3) and the eight cones of (3.4), and these are a resolution of the singularity $\frac{1}{17}(1, 1, 6, 9)$.

If we chose $p_3e_2p_4p_6$ instead of $p_1e_2p_4p_6$, we would obtain a resolution, but only after making more choices of neighbours.

3.2 Setup

The algorithm works by first computing the cones which sit at the faces of the junior simplex. The idea is first to tile the faces and then to work inwards using the `ConeChains`. We start by finding the `FaceCones`. There cannot be more than one of these for each face $e_ie_je_k$. This is because, as the group is cyclic, we can't have $\frac{1}{r}(a, b, c, 1)$ and $\frac{1}{r}(d, e, f, 1)$ unless $a = d, b = e$, and $c = f$, so we can't have two distinct basic cones $e_1e_2e_3p_a$ and $e_1e_2e_3p_d$.

We find the junior points lying on the faces of the junior simplex. We then find cones in `CrepantCones` whose faces form part of the junior simplex. We call this set `FaceCones`. It may contain more than four cones, and we may have to make a choice between the cones.

Lemma 3.2.1. *For cyclic groups of order $r \geq 4$ there are at least four* `FaceCones`.

Proof. We begin by proving that there are four cones of the form $e_ie_je_kp$ if and only if there are no junior points lying on the faces of the junior simplex.

Suppose there is no crepant cone of the form $e_1e_2e_3p$. We know that if there were a point $p = \frac{1}{r}(d_1, d_2, d_3, 1)$ in the unit box inside the lattice $L = \mathbb{Z}^4 + \frac{1}{r}(a_1, a_2, a_3, a_4) \cdot \mathbb{Z}$ then p would have to be a junior point. Thus, since no such p exists, r and a_4 must have a common factor, say s. Let $d = \frac{r}{s}$ and let

$$\alpha_i = da_i \mod r$$

for $1 \leq i \leq 4$. Now $\alpha_4 = 0$ and

$$\alpha_1 + \alpha_2 + \alpha_3 = \text{either } r \text{ or } 2r.$$

Now $(a_4 - 1)da_i = a_4da_i - da_i = r - \alpha_i \mod r$ for all i. So

$$r - \alpha_1 + r - \alpha_2 + r - \alpha_3 = \text{either } r \text{ or } 2r.$$

Thus, either $\frac{1}{r}(\alpha_1, \alpha_2, \alpha_3, 0)$ has age 1 or $\frac{1}{r}(r - \alpha_1, r - \alpha_2, r - \alpha_3, 0)$ has age 1. So we have at least one junior point on the face $e_1e_2e_3$.

Conversely, suppose p is a junior point on the face $e_1e_2e_3$, so that we can write $p = \frac{1}{r}(\alpha_1, \alpha_2, \alpha_3, 0)$. Then the cone $e_1e_2e_3q$ cannot be basic for any junior point q, since the cones pe_2e_3q, e_1pe_3q and e_1e_2pq are all contained in it and are generated by vectors in the lattice.

We prove that each cone has at most one face contained in a face of the junior simplex. Suppose not. Let $i, j, k, l \in \{1, 2, 3, 4\}$ be distinct. Without loss of generality we can assume $i = 1,\ j = 2,\ k = 3,\ l = 4$. Up to symmetry, there are the following options:

1. There is one vertex at e_1. The other three vertices lie on the edges e_1e_2, e_1e_3, e_1e_4.

2. There is one vertex at each of e_1 and e_2. The other two vertices lies on the edges e_1e_3, e_1e_4.

3. There is one vertex at e_1. A vertex lies on each of the edges e_1e_2, e_1e_3. One vertex lies on the face $e_1e_2e_4$.

4. There is one vertex at each of e_1 and e_2. One vertex lies on the edge joining e_1 to e_3. One vertex lies on the face $e_1e_2e_4$.

5. There is one vertex at e_1, one vertex on an edge e_1e_2 and one vertex on each of the faces $e_1e_2e_3$, $e_1e_2e_4$.

6. There is one vertex at each of e_1 and e_2, and one vertex on each of the faces $e_1e_2e_3$, $e_1e_2e_4$.

7. There are two vertices on the line e_1, e_2, and one vertex on each of the faces $e_1e_2e_3$, $e_1e_2e_4$.

8. There is a vertex at each of e_1, e_2, e_3, and one vertex on the edge e_1e_4.

9. There is a vertex at each of e_1, e_2, e_3, and one vertex on the face $e_1e_2e_4$.

In the first six cases, the vertices of the cone will give us, at worst, an upper triangular matrix:

$$\begin{vmatrix} r & 0 & 0 & 0 \\ r-a & a & 0 & 0 \\ b_1 & b_2 & b_3 & 0 \\ c_1 & c_2 & 0 & c_4 \end{vmatrix}.$$

This is a basic cone if $ab_3c_4 = r^2$.

If these points come from a cyclic group action, say by $\frac{1}{r}(a_1, a_2, a_3, a_4)$, then we have:

$$\begin{array}{lll} \alpha a_2 = a & \beta a_2 = b_2 & \gamma a_2 = c_2 \\ \alpha a_3 = \lambda_1 r & \beta a_3 = b_3 & \gamma a_3 = \nu_1 r \\ \alpha a_4 = \lambda_2 r & \beta a_4 = \mu_1 r & \gamma a_4 = c_4, \end{array}$$

with $\alpha, \beta, \gamma, \lambda_1, \lambda_2, \mu_1, \nu_1$ strictly positive integers. Thus

$$\begin{aligned} r^2 &= ab_3c_4 \\ &= \alpha a_2 \beta a_3 \gamma a_4 \\ &= \alpha a_2 \beta a_4 \gamma a_3 \\ &= \mu_1 \nu_1 \alpha a_2 r^2. \end{aligned}$$

This is a contradiction, so the first six cases cannot happen.

Consider a cone of the form

$$\begin{vmatrix} r-a & a & 0 & 0 \\ r-b & b & 0 & 0 \\ c_1 & c_2 & c_3 & 0 \\ d_1 & d_2 & 0 & d_4 \end{vmatrix}.$$

Without loss of generality, assume $b > a$. This is a basic cone if

$$(r-a)bc_3d_4 - a(r-b)c_3d_4 = r^3.$$

If these points come from a cyclic group action, say by $\frac{1}{r}(a_1, a_2, a_3, a_4)$, then we have:

$$\begin{aligned} \alpha a_3 &= c_3 & \beta a_3 &= \mu r \\ \alpha a_4 &= \lambda r & \beta a_4 &= d_4, \end{aligned}$$

with $\alpha, \beta, \lambda, \mu$ strictly positive integers. Hence

$$\begin{aligned} r^3 &= (r-a)bc_3d_4 - a(r-b)c_3d_4 \\ &= r(b-a)c_3d_4 \\ &= (b-a)\lambda\mu r^3. \end{aligned}$$

This disproves the seventh case.

Consider a cone of the form

$$\begin{vmatrix} r & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & r & 0 \\ r-d & 0 & 0 & d \end{vmatrix}.$$

We must have $d = 1$ if this is to be a basic cone.

If these points come from a cyclic group action, say by $\frac{1}{r}(a_1, a_2, a_3, a_4)$, then we have:

$$\alpha a_1 = \kappa r - 1 \quad \alpha a_2 = \lambda r \quad \alpha a_3 = \mu r \quad \alpha a_4 = \nu r + 1.$$

Thus a_2 and r share a common factor, and a_1, a_4 are coprime to r.

Write $r = ab$ and $a_2 = ac$ such that $\mathrm{hcf}(b, c) = 1$. Then $\alpha a_2 = \alpha ac = \lambda ab$, so α divides b. Hence $\alpha a_4 \neq 1 \mod r$ since α and r are not coprime.

The ninth case is similar.

Thus for each of the four faces of the junior simplex there is at least one cone, one of whose faces is part of that face and none of its other faces are part of another face of the junior simple. □

This lemma means that if we have four face cones of the form

$$p_1e_2e_3e_4, \quad e_1p_2e_3e_4, \quad e_1e_2p_3e_4, \quad e_1e_2e_3p_4$$

we do not need to search for any more face cones.

3.3 Face pieces

The code consists of two main functions HasCrepRes and FindResn. Algorithm 1 shows how HasCrepRes works and sets up the data required for FindResn, which will be described in Algorithm 2. Both algorithms initiate tree searches. The function HasCrepRes finds a tiling of the faces of the junior simplex, then calls FindResn to complete the search for a crepant resolution with these initial conditions. If FindResn fails to find a resolution the initial face tiling is changed, and the algorithm runs until a resolution is found or all possible face tilings have been tested. The algorithms here are pseudocode; the Magma code is available at `http://www.warwick.ac.uk/staff/S.E.Davis/Thesis/ResolutionAlgorithm.m`

The input for HasCrepRes is the singularity $\frac{1}{r}(a_1, a_2, a_3, a_4)$ expressed as `r,` $[a_1, a_2, a_3, a_4]$. We begin by setting up all the objects we need. We make the list, `CrepantCones`, of crepant cones; the list, `OverlapGraph`, of which cones overlap, the list, `AdjacencyGraph`, of neighbours at each face; the list, `ConeChains`, of cones forced by each cone; the list, `Juniors`, of junior points; the list, `FaceCones`, of cones which sit on the faces of the junior simplex; and `ConeChainOverlapGraph`, the table of which `ConeChains` overlap.

The full algorithm contains extra optimisation: after we have created `ConeChainOverlapGraph` we restrict ourselves to working only with the subset `AllowedCones` of `CrepantCones` consisting of cones whose `ConeChains` do not overlap themselves.

We start the repeat loop (Algorithm 1, line 9) with `ChosenFaces = [ ]`. We will choose cones from `FaceCones` one at a time. This will put restrictions on which cones we may choose next, so we create a list `PossibleFaces`, which is initially equal to `FaceCones`. We will need to keep track of the order in which we

Algorithm 1 HasCrepRes

```
function HasCrepRes( r, [a1, a2, a3, a4] )
    CrepantCones := MakeCrepantCones(r, [a1, a2, a3, a4])
    OverlapGraph := MakeOverlapGraph(CrepantCones)
    AdjacencyGraph := MakeAdjacencyGraph(CrepantCones)
    ConeChains := MakeConeChains(CrepantCones, AdjacencyGraph)
    Juniors := FindJuniors(r, [a1, a2, a3, a4])
    FacePieces := FindFaces(Juniors,CrepantCones)
    ConeChainOverlapGraph := MakeMatrix(ConeChains,OverlapGraph)
    repeat
        ChosenFaces := [ ]
        FaceSteps := [[ ChosenFaces, FacePieces ]]
        run MakeFaceTiling(FaceSteps, ConeChains)
        ChosenCones := the union of the ConeChains of cones in FaceTiling
        if length(ChosenCones) ≠ r then
            run FindResn(ChosenCones)
                        ▷ Look for a tiling of the interior of the junior simplex.
        end if
        if we haven't found a resolution then
            run UndoLastFace(FaceSteps)
                        ▷ Undo the last choice of face we made.
        end if
    until we have a resolution or there are no choices of FaceCones left
return False if no resolution exists,
    True and a Resolution otherwise.
end function
```

have chosen cones for the `FaceTiling`; we do this using the list `FaceSteps`:

```
FaceSteps =
  [[ChosenFaces1,PossibleFaces1],[ChosenFaces2,PossibleFaces2],...].
```

We want to choose `FaceCones` which tile the faces of the junior simplex. If `FaceCones` contains exactly four cones there is only one way to do this. We may, however, have to make a choice. This is done by the following function:

```
1:  function MakeFaceTiling(FaceSteps, ConeChains)
2:      n := length(FaceSteps)
3:      ChosenFaces := FaceSteps[n,1]
4:      PossibleFaces := FaceSteps[n,2]
                                        ▷ Look at last entries of FaceSteps.
5:      while PossibleFaces ≠ [] do
6:          Append [ChosenFaces, PossibleFaces] to FaceSteps
                                        ▷ Save the last choice we made.
7:          Cone1 := PossibleFaces[1]
                                        ▷ Choose the first cone.
8:          NewFaces := ConeChains[Cone1] ∩ PossibleFaces
9:          Append NewFaces to ChosenFaces
                                        ▷ Add in all PossibleFaces forced by Cone1.
10:         for Cone in ChosenFaces do
11:             Exclude from PossibleFaces any cone whose ConeChain overlaps
    with that of Cone
12:         end for
13:     end while
14:     FaceSteps := FaceSteps ∪ [FaceTiling, PossibleFaces]
15: return FaceSteps
16: end function
```

Here, `FaceSteps[n,1]` means take the first entry of the `n`th row of `FaceSteps`.

At line 6 we save the last choice we made: we are saving the choices we made in the last iteration of the while loop. On the first iteration we save our original data for a second time, so

```
FaceSteps =
  [[ChosenFaces1,PossibleFaces1],[ChosenFaces2,PossibleFaces2]].
```

with

```
ChosenFaces1 = ChosenFaces2 = [ ]
```

and

```
FaceCones1=FaceCones2=FaceCones.
```

Since we do not save the last choice at the end of the while loop we save once we have exited the loop (line 14)

When `PossibleFaces` becomes empty, `FaceTiling` is the last set of `ChosenFaces` in `FaceSteps`. The cones of `FaceTiling` may force some interior cones, so we take the union of the `ConeChains` of all cones in `FaceTiling`. We call this list `ChosenCones`.

If `ChosenCones` contains r cones we have a resolution and there is no more to do. If not, we run the function FindResn, which is described in Algorithm 2. Again, if this returns a resolution we are done. If not, we see if it is possible to choose a different `FaceTiling`. The next algorithm uses `FaceSteps` to undo the last choice we made whilst setting up `FaceTiling`:

```
function UndoLastFaces(FaceSteps)
    n := length(FaceSteps)
    LastFaceTiling := FaceSteps[n,1]
    PenultimateFaceTiling := FaceSteps[n-1,1]
    LastPossibleFaces := FaceSteps[n,2]
    PenultimatePossibleFaces := FaceSteps[n-1,2]
    JustAdded := PenultimateFaceTiling ∩ LastFaceTiling
                                        ⊳ Compare the newly added cones
    while JustAdded is empty do
                                        ⊳ There are no newly added cones
                                        ⊳ delete the last entry of FaceSteps
        n := n-1
                                        ⊳ and reset the variables.
        LastFaceTiling := FaceSteps[n,1]
        PenultimateFaceTiling := FaceSteps[n-1,1]
        LastPossibleFaces := FaceSteps[n,2]
        PenultimatePossibleFaces := FaceSteps[n-1,2]
```

```
14:         JustAdded := PenultimateFaceTiling ∩ LastFaceTiling
15:     end while
                                        ▷ There are newly added cones
16:     Delete FaceSteps[n]
                                  ▷ delete the last entry of FaceSteps
17:     Delete JustAdded[1] from PenultimatePossibleFaces
18:     FaceSteps[n-1] := [PenultimateFaceTiling,
                              PenultimatePossibleFaces]
        ▷ remove the first cone of JustAdded from PenultimatePossibleFaces.
19: return FaceSteps
20: end function
```

Magma does not rearrange the order of the cones in `LastFaceTiling` so the first cone of `JustAdded` (see line 7) is the cone the choice of which forced all other cones in `JustAdded`. (Recall that we update `FaceSteps` after every choice, so all cones in `JustAdded` are forced by this choice.) If `JustAdded` is empty no cones were chosen in the last step, so we go back to a step where cones were added (if one exists). The function deletes the last entry of `FaceSteps` and resets the variables. Once we have reached a step where `JustAdded` is not empty the last entry of `FaceSteps` is deleted and the chosen cone is removed from `PenultimatePossibleFaces`, so that it cannot be chosen at this step again.

Note that when we calculate `JustAdded` in Magma we don't change the order of the cones. Thus, the first entry of `JustAdded` is the cone we chose the last time we made a choice. Any other cones in `JustAdded` were forced by the first cone. By deleting the first entry of `JustAdded` from `PenultimatePossibleFaces` we prevent ourselves from choosing the same cone again, unless we change our earlier choices of cones.

3.4 Find a resolution

The function FindResn is summarised in Algorithm 2. It is also demonstrated in the flowchart of Figure 2.

On line 10 the notation `Saved := Saved` $\cup$ `[[Cones,Choices]]` means append `[[Cones,Choices]]` to the end of `Saved` without changing the order of the previous entries of `Saved`.

Algorithm 2 FindResn

```
function FindResn(ChosenCones)
    Choices := MakeChoices(ChosenCones)
    Cones := ChosenCones
    Saved := [[Cones, Choices]]
    while Choices is not empty do
        while no Choice in Choices is empty and length(Cones) < r cones do
            while there is a Choice in Choices which contains exactly one cone do
                run MakeUniqueChoice with that Choice
            end while
            Saved := Saved ∪ [[Cones,Choices]]
            Cones, Choices := MakeAChoice(Cones,Choices)
            if Choices is empty then
                Choices := MakeChoices(Cones)
                Saved:= Saved ∪ [[Cones,Choices]]
            end if
        end while
        if there are not r cones in Cones then
            Saved := UndoLastStep(Saved)
        end if
    end while
return Saved
end function
```

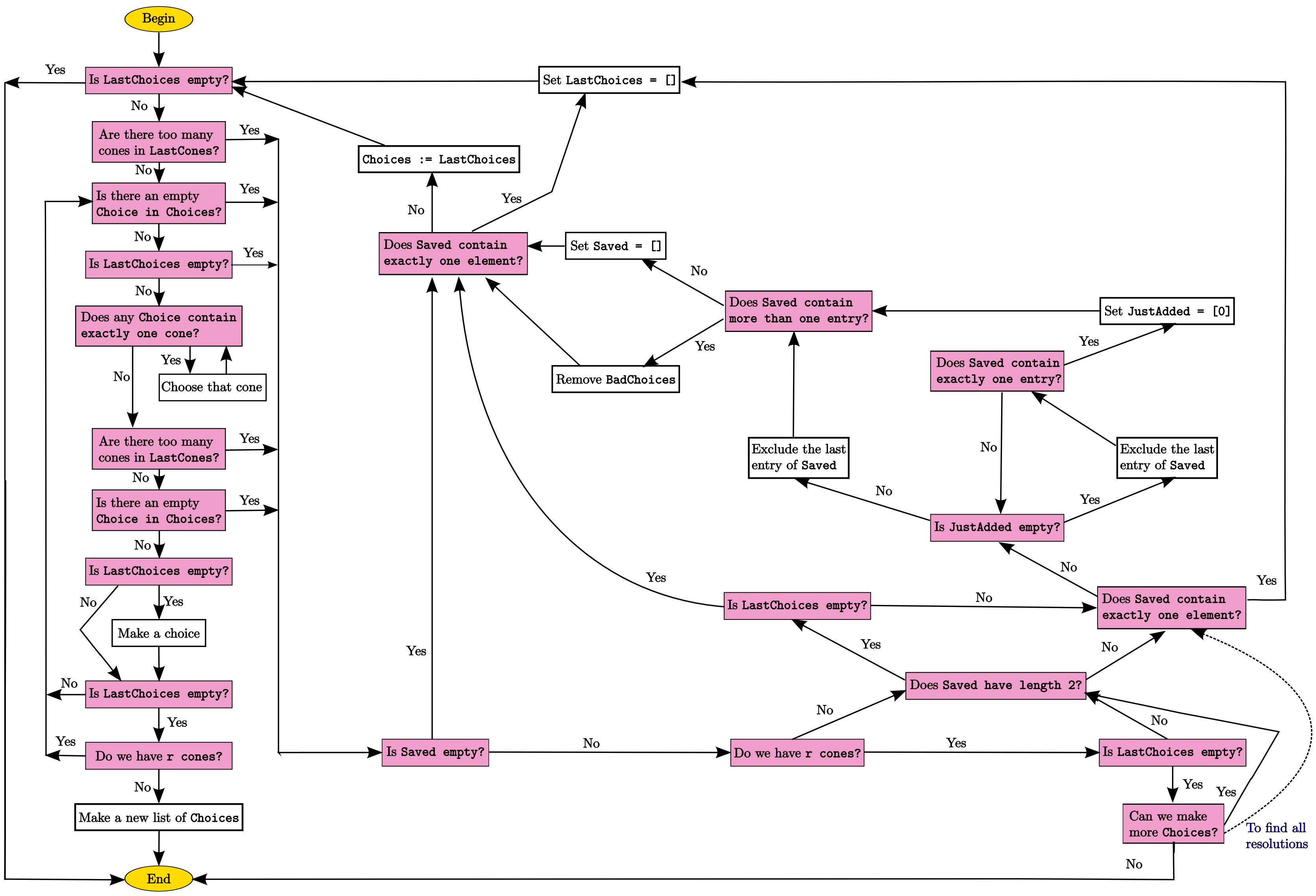

Figure 3.2: Flowchart for the FindResn algorithm

FINDRESN is called by HASCREPRES, which has already set up a `FaceTiling` of the junior simplex and taken any cones which are forced by this tiling. The idea now is to work inwards, making the choice of which cone to add at each face where there is no neighbouring cone. The first step is to find the available choices at every face. This is done by the function MAKECHOICES.

The input for MAKECHOICES is `ChosenCones`. Suppose a cone has no neighbouring cone in `ChosenCones` at one of its faces, but has at least one neighbouring cone from `CrepantCones` at the same face. We must add one of these neighbouring cones to the triangulation. The function MAKECHOICES finds all faces where there is no neighbour in `ChosenCones` and saves the set of possible neighbours at each of these faces as a list, which we refer to as an `n-choice`, where `n` is the number of possible neighbours in that list. The list of all `n-choices` is called `Choices`. Before MAKECHOICES returns `Choices` it removes any overlapping cones from `Choices`.

Having found which cones may be neighbours to our `ChosenCones`, we define (line 4) a list of pairs called `Saved`. To start with `Saved` is just the pair consisting of the set `ChosenCones` and the set `Choices`, which was the output of MAKECHOICES.

We must choose a cone from each set in `Choices`. If there are any `0-choices` we are not able to find a resolution so we have to make a change (at line 6 we skip to line 16). Any `1-choices` are forced cones. If there are any such choices we enter the **while** loop at line 7 and the function MAKEUNIQUECHOICE is called. This function takes the cone, C, in the first `1-choice` of `Choices` and adds C and its `ConeChain` to `Cones`. Every time a cone, D belonging to the `ConeChain` of C appears in an `n-choice` of `Choices` that `n-choice` is excluded, as these choices have just been made. Any cone (other than D) in this `n-choice` now cannot appear in the triangulation. We call the set of all such cones `NotAllowed`.

The function MAKEUNIQUECHOICE goes on to remove all the cones (and their `ConeChains`) in `NotAllowed` from `Choices`. Note that this may lead to an empty `n-choice` in `Choices`.

The `ConeChain` of C has been added to `Cones`. Now no cone whose `ConeChain` overlaps with the `ConeChain` of C can be chosen. The function MAKEUNIQUE-CHOICE removes any such cone from the sets of `Choices`. Again this may leave an empty n-choice in `Choices`. The function MAKEUNIQUECHOICE is now complete and we return to line 7.

If we have no `1-choices` we proceed to line 9. We must decide between the elements in the `n-choices`. We do this using MAKEACHOICE. This function takes the first `n-choice` in `Choices` and takes the first cone, C, in it. It proceeds as in MAKEUNIQUECHOICE: it adds in the `ConeChain` of C, removes any `n-choices` it has made by doing this and removes any overlapping cones.

If `Choices` is empty (line 12) we run MAKECHOICES again to make a new set of choices if this is possible.

If we have made the wrong choice (we failed to find a resolution because we ran out of choices or were unable to make a choice at a certain face) we enter the **if** statement at line 17. The function UNDOLASTSTEP deletes the last entry and edits the penultimate entry of `Saved`. We may edit the second entry of `Saved`, but as this was originally the same as the first, it is deleted when the tree search terminates.

We will refer to the entries of the last pair in `Saved` as `LastCones` and `LastChoices`. Similarly `PenultimateCones` and `PenultimateChoices` are the entries of the penultimate pair in `Saved`.

The function UNDOLASTSTEP is very similar to the UNDOLASTFACES function of HASCREPRES. We delete the last entry of `Saved` until we get to a point where we added cones to `LastCones`. We delete the cone we chose from the penultimate list of choices, and delete the last entry of `Saved`. This leaves us in a position to make a new choice of cone.

The algorithm then terminates if we find a resolution or when we have explored all possible arrangements of cones. The variable `Saved` is returned in both cases.

3.5 Justification of algorithm

We will now justify that the algorithm terminates and that it finds a crepant resolution if one exists.

Let T be the set of all triangulations of $\{a_1, \ldots, a_n, e_1, e_2, e_3, e_4\}$, the set of junior points including vertices.

Claim 3.5.1. *Every triangulation Δ of T can be expressed as the union of cone chains.*

Proof. Suppose not. Suppose Δ cannot be expressed as a union of cone chains. Then either:

- There exists one cone which does not belong to a cone chain. This is impossible since every cone has a cone chain;

- An element of a cone chain is missing. If this were the case, one cone in the cone chain would have been replaced with a different cone with a common face. However, cone chains are the set of forced neighbours, so there are no further cones with this face.

Thus every triangulation can be expressed as the union of cone chains. □

A problem would occur if we chose a set of cone chains whose union was not a triangulation. We show that this cannot happen by first showing that it is not possible to pick two cones in such a way that we cannot obtain a triangulation.

Since the cones are basic, it is not possible for two cones to meet at a vertex which is not a vertex of both of the cones. This means they can't meet in part of an edge either:

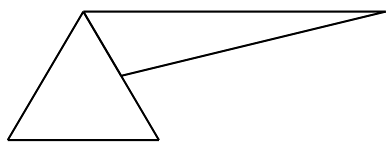

Figure 3.3: Not a triangulation: two cones meeting in a subset of an edge

It is, however, possible for two cones to meet at a subset of a face, or a subset of volume 3.

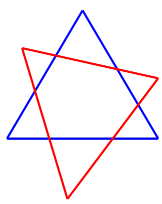

Figure 3.4: Two cones meeting in a subset of a face

We can have the arrangement of Figure 3.4 with the missing vertex of the red cone coming out of the page and the missing vertex of the blue cone going into the page. If this happens we would not be able to find neighbours to the cones at the faces drawn, as the possible neighbouring cones of the blue cone would overlap with the red cone and vice versa. If this happened during the algorithm it would undo some of the previous steps until either the blue cone or the red cone had been discarded.

The algorithm would not choose this arrangement with both the missing vertices coming out of the page. This would mean the red and blue cone intersect with nonzero volume which is not allowed (this is the same situation as both the red and the blue cone missing the same vertex).

The algorithm does not allow us to pick cones which meet in a subset of volume three, as we use the overlap data to check this at each step. Now we can consider the cone chains.

Given a single cone chain C, if a pair of cones of C intersect with dimension greater than three then C cannot appear as part of any resolution (as this contradicts the definition of triangulation). We will exclude any cone chain where this happens.

Any pair of cone chains are allowed to meet in an edge, a face or in a whole cone. If their intersection was something other than one of these we would not have a triangulation. As we have just seen, the algorithm does not allow cones to meet in a subset of volume three, and cones cannot meet in part of an edge. Thus the only possible bad intersection would be if a pair of cones, one from each cone chain, met in a subset of a face, as in Figure 3.4. But this would again lead to there being two faces in the resolution where it would be impossible to find neighbours.

Thus if we have a union of cone chains which contains r distinct basic cones, none of which have a bad intersection, then we must have a resolution. The basic cones can be thought of as simplices of relative volume 1, where the junior simplex has relative volume r. Thus these r basic cones must triangulate the junior simplex.

The algorithm works by first finding a face tiling and then working inwards. In fact, the order in which we choose the cones does not matter, as we check every combination until a resolution is found. The use of the face tiling gives a smaller set of cones to start from. Any resolution must certainly include a face tiling.

The algorithm must find a resolution if one exists as we search through every possible combination of cones.

3.6 Examples

Example 3.6.1. Consider the quotient singularity $\frac{1}{7}(1,1,1,4)$. This was discussed in Example 3.1.2. The example has two junior points and there are exactly seven basic cones.

```
> r:=7;A:=[1,1,1,4];
> Juniors:=FindJuniors(r,A);
```

```
> Juniors;
[
    [ 1, 1, 1, 4 ],
    [ 2, 2, 2, 1 ]
]
> CrepantCones := MakeCrepantCones(r,A);
> #CrepantCones;
7
> CrepantCones;
[
    [
        [ 1/7, 1/7, 1/7, 4/7 ],
        [ 2/7, 2/7, 2/7, 1/7 ],
        [ 1, 0, 0, 0 ],
        [ 0, 1, 0, 0 ]
    ],
    [
        [ 1/7, 1/7, 1/7, 4/7 ],
        [ 2/7, 2/7, 2/7, 1/7 ],
        [ 1, 0, 0, 0 ],
        [ 0, 0, 1, 0 ]
    ],
    [
        [ 1/7, 1/7, 1/7, 4/7 ],
        [ 2/7, 2/7, 2/7, 1/7 ],
        [ 0, 1, 0, 0 ],
        [ 0, 0, 1, 0 ]
    ],
    [
        [ 1/7, 1/7, 1/7, 4/7 ],
        [ 1, 0, 0, 0 ],
        [ 0, 1, 0, 0 ],
        [ 0, 0, 0, 1 ]
    ],
    [
        [ 1/7, 1/7, 1/7, 4/7 ],
```

```
        [ 1, 0, 0, 0 ],
        [ 0, 0, 1, 0 ],
        [ 0, 0, 0, 1 ]
    ],
    [
        [ 1/7, 1/7, 1/7, 4/7 ],
        [ 0, 1, 0, 0 ],
        [ 0, 0, 1, 0 ],
        [ 0, 0, 0, 1 ]
    ],
    [
        [ 2/7, 2/7, 2/7, 1/7 ],
        [ 1, 0, 0, 0 ],
        [ 0, 1, 0, 0 ],
        [ 0, 0, 1, 0 ]
    ]
]
```

It is not hard to see that the face tiling is the last four cones.

```
> FacePieces := FindFaces(Juniors,CrepantCones);
> FacePieces;
[ 4, 5, 6, 7 ]
```

In fact the cone chain of cone 4 is all of CrepantCones.

```
>AdjacencyGraph :=  MakeAdjacencyGraph(CrepantCones);
> ConeChains := MakeConeChains(CrepantCones,AdjacencyGraph);
> ConeChains[4];
[ 4, 1, 2, 3, 5, 6, 7 ]
> OverlapGraph := MakeOverlapGraph(CrepantCones);
> ConeChainOverlapGraph := MakeConeChainOverlaps(ConeChains,
                                                  OverlapGraph);
> AllowedCones := [1..#CrepantCones];
>       for i in [1..#ConeChainOverlapGraph] do
for>            if ConeChainOverlapGraph[i,i] then
for|if>                        Exclude(~AllowedCones,i);
for|if>                        Exclude(~FacePieces,i);
```

```
for|if>                 end if;
for>    end for;
> #AllowedCones;
7
```

Since all of the basic cones are allowed, they are the unique crepant resolution of $\frac{1}{7}(1,1,1,4)$.

Example 3.6.2. A crepant resolution of the quotient singularity $\frac{1}{17}(1,3,3,10)$ was computed via barycentric subdivision in Example 2.1.1. This Magma output shows how the algorithm works on this example.

```
> r:=17;A:=[1,3,3,10];
> Juniors:=FindJuniors(r,A);
> Juniors;
[
    [ 1, 3, 3, 10 ],
    [ 2, 6, 6, 3 ],
    [ 6, 1, 1, 9 ],
    [ 7, 4, 4, 2 ],
    [ 12, 2, 2, 1 ]
]
> CrepantCones := MakeCrepantCones(r,A);
> #CrepantCones;
33
> FacePieces := FindFaces(Juniors,CrepantCones);
> FacePieces;
[ 33, 30, 20, 31 ]
```

There are 33 basic cones, four of which are face cones, so form a face tiling of the junior simplex.

```
>AdjacencyGraph :=  MakeAdjacencyGraph(CrepantCones);
> ConeChains := MakeConeChains(CrepantCones,AdjacencyGraph);
> ConeChains[33];
[ 33, 32, 23, 5, 20 ]
```

In this example the cone chains of face cones are short. In fact the face tiling forces only 7 cones:

```
> OverlapGraph := MakeOverlapGraph(CrepantCones);
> ConeChainOverlapGraph := MakeConeChainOverlaps(ConeChains,
                                                  OverlapGraph);
> AllowedCones := [1..#CrepantCones];
>       for i in [1..#ConeChainOverlapGraph] do
for>            if ConeChainOverlapGraph[i,i] then
for|if>                         Exclude(~AllowedCones,i);
for|if>                         Exclude(~FacePieces,i);
for|if>                 end if;
for>    end for;
> #AllowedCones;
33
> ChosenFaces := FacePieces;
> ChosenCones := MakeForced(ConeChains,ChosenFaces);
> #ChosenCones;
7
> ChosenCones;
[ 33, 23, 5, 30, 20, 31, 32 ]
```

Since there are not yet 17 cones, more cones must be chosen. The faces of cones in ChosenCones where there is no neighbour in ChosenCones are considered. A search for possible neighbours at these faces yields a set of choices:

```
> Choices := MakeChoices(CrepantCones, ChosenCones,AdjacencyGraph,
                          ConeChainOverlapGraph, AllowedCones);
> Choices;
[
    [ 19, 29 ],
    [ 18, 28 ],
    [ 4, 22 ],
    [ 3, 21 ],
    [ 2, 4 ],
    [ 1, 3 ],
    [ 12, 26 ],
    [ 10, 28 ],
    [ 13, 17 ],
    [ 12, 16 ],
```

```
    [ 13, 27 ],
    [ 11, 29 ],
    [ 15, 25 ],
    [ 14, 24 ]
]
```

Cones 19 and 29 are neighbours of cone 33:

```
> CrepantCones[33];
[
    [ 12/17, 2/17, 2/17, 1/17 ],
    [ 1, 0, 0, 0 ],
    [ 0, 1, 0, 0 ],
    [ 0, 0, 1, 0 ]
]
> CrepantCones[19];
[
    [ 1/17, 3/17, 3/17, 10/17 ],
    [ 12/17, 2/17, 2/17, 1/17 ],
    [ 1, 0, 0, 0 ],
    [ 0, 0, 1, 0 ]
]
> CrepantCones[29];
[
    [ 6/17, 1/17, 1/17, 9/17 ],
    [ 12/17, 2/17, 2/17, 1/17 ],
    [ 1, 0, 0, 0 ],
    [ 0, 0, 1, 0 ]
]
>
```

Exactly one of these cones must belong to the resolution. The algorithm will pick cone 19 first.

```
> cone:=Choices[1,1];
> cone;
19
> ConeChains[cone];
```

```
[ 19, 33, 11, 13, 12, 15, 14, 3, 23, 4, 5, 18, 30, 31, 20, 32, 10 ]
> #ConeChains[cone];
17
> &and[&and[OverlapGraph[i,j] : j in [i+1..17]]: i in [1..16]];
false
```

The cone chain of cone 19 contains 17 cones. These cones do not overlap so they form a tiling of the junior simplex, and thus correspond to a resolution of the quotient singularity $\frac{1}{17}(1,3,3,10)$.

If instead cone 29 is picked, its cone chain contains only 9 cones, so more choices must be made. However first the list of choices must be updated. It is clear now cones 19 and 11 cannot appear in a resolution containing cone 29, since these both appear in a set in choices that also contains 19. Since cone 28 is in the cone chain of cone 29, cones 18 and 10 must also be removed from choices.

```
> cone:=Choices[1,2];
> cone;
29
> ConeChains[cone];
[ 29, 33, 28, 30, 31, 32, 23, 5, 20 ]
> #ConeChains[cone];
9
> Choices2,NotAllowed:=MakeAChoice(Choices,cone,ConeChains);
> Choices2;
[
    [ 4, 22 ],
    [ 3, 21 ],
    [ 2, 4 ],
    [ 1, 3 ],
    [ 12, 26 ],
    [ 13, 17 ],
    [ 12, 16 ],
    [ 13, 27 ],
    [ 15, 25 ],
    [ 14, 24 ]
]
> NotAllowed;
[ 19, 18, 10, 11 ]
```

After running the algorithm further this leads to a resolution.

```
> C,D:=HasCrepResns(r,A);
> #D;
40
> for i in [1..#D] do
for> D[i]:=Sort(D[i]);
for> end for;
> SetToSequence(SequenceToSet(D));
[
    [ 3, 4, 5, 6, 7, 12, 13, 20, 23, 24, 25, 28, 29, 30, 31, 32, 33 ],
    [ 3, 4, 5, 10, 11, 12, 13, 14, 15, 18, 19, 20, 23, 30, 31, 32, 33 ],
    [ 3, 4, 5, 8, 9, 12, 13, 14, 15, 20, 23, 28, 29, 30, 31, 32, 33 ],
    [ 3, 4, 5, 16, 17, 20, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33 ],
    [ 1, 2, 5, 12, 13, 20, 21, 22, 23, 24, 25, 28, 29, 30, 31, 32, 33 ]
]
```

The algorithm shows that the are five different crepant resolutions of $\frac{1}{17}(1,3,3,10)$. Choosing cone 19 gave the second of these, whereas cone 29 belongs to each of the other four resolutions.

Example 3.6.3. Consider the quotient singularity $\frac{1}{18}(1,1,3,13)$. In this example 3 divides 18, so there is a junior point lying on a face of the junior simplex. Thus there are more face pieces.

```
> r:=18;A:=[1,1,3,13];
> Juniors:=FindJuniors(r,A);
> Juniors;
[
    [ 1, 1, 3, 13 ],
    [ 2, 2, 6, 8 ],
    [ 3, 3, 9, 3 ],
    [ 6, 6, 0, 6 ],
    [ 7, 7, 3, 1 ]
]
> CrepantCones := MakeCrepantCones(r,A);
> #CrepantCones;
32
```

```
> FacePieces := FindFaces(Juniors,CrepantCones);
> FacePieces;
[ 11, 14, 15, 29, 30, 9, 31, 32, 10 ]
```

These face pieces overlap so the algorithm chooses a face tiling:

```
> ChosenFaces;
[ 11, 15, 14, 30, 31, 10, 32 ]
> CrepantCones[ChosenFaces];
[
    [
        [ 1/18, 1/18, 1/6, 13/18 ],
        [ 1/3, 1/3, 0, 1/3 ],
        [ 0, 1, 0, 0 ],
        [ 0, 0, 0, 1 ]
    ],
    [
        [ 1/18, 1/18, 1/6, 13/18 ],
        [ 0, 1, 0, 0 ],
        [ 0, 0, 1, 0 ],
        [ 0, 0, 0, 1 ]
    ],
    [
        [ 1/18, 1/18, 1/6, 13/18 ],
        [ 1, 0, 0, 0 ],
        [ 0, 0, 1, 0 ],
        [ 0, 0, 0, 1 ]
    ],
    [
        [ 1/3, 1/3, 0, 1/3 ],
        [ 7/18, 7/18, 1/6, 1/18 ],
        [ 1, 0, 0, 0 ],
        [ 0, 0, 0, 1 ]
    ],
    [
        [ 1/3, 1/3, 0, 1/3 ],
        [ 7/18, 7/18, 1/6, 1/18 ],
        [ 0, 1, 0, 0 ],
```

```
        [ 0, 0, 0, 1 ]
    ],
    [
        [ 1/18, 1/18, 1/6, 13/18 ],
        [ 1/3, 1/3, 0, 1/3 ],
        [ 1, 0, 0, 0 ],
        [ 0, 0, 0, 1 ]
    ],
    [
        [ 7/18, 7/18, 1/6, 1/18 ],
        [ 1, 0, 0, 0 ],
        [ 0, 1, 0, 0 ],
        [ 0, 0, 1, 0 ]
    ]
]
```

The algorithm continues as in the previous examples, by finding the forced cones and then searching for a tiling of the interior of the junior simplex. The algorithm will choose a different face tiling if this one does not lead to a tiling of the junior simplex, or if it is searching for all resolutions.

```
[
    [ 1, 2, 5, 6, 10, 11, 14, 15, 16, 17, 20, 21, 25, 26, 27, 28, 29, 32 ],
    [ 1, 2, 5, 6, 10, 11, 14, 15, 18, 19, 20, 21, 22, 23, 27, 28, 29, 32 ],
    [ 3, 4, 5, 6, 12, 13, 14, 15, 18, 19, 20, 21, 27, 28, 29, 30, 31, 32 ],
    [ 3, 4, 5, 6, 7, 8, 10, 11, 14, 15, 18, 19, 20, 21, 27, 28, 29, 32 ]
]
```

Four different tilings have been found, only one of which contains the face tiling above.

Example 3.6.4. The quotient singularity $\frac{1}{13}(1, 1, 4, 7)$ does not have a crepant resolution.

```
> HasCrepRes(13,[1,1,4,7]);
false []
```

It is easy to see why this happens; there are only 11 crepant cones.

```
> r:=13; A:=[1,1,4,7];
> Juniors := FindJuniors(r,A);
> Juniors;
[
    [ 1, 1, 4, 7 ],
    [ 2, 2, 8, 1 ],
    [ 4, 4, 3, 2 ]
]
> CrepantCones := MakeCrepantCones(r,A);
> #CrepantCones;
11
```

Now consider the the age 2 points of the lattice $\mathbb{Z}^4 + \frac{1}{13}(1,1,4,7)$.

```
> Pts:=Pts(r,A);
> Age2:=[P : P in Pts | &+P eq 2*r];
> Age2;
[
    [ 3, 3, 12, 8 ],
    [ 5, 5, 7, 9 ],
    [ 6, 6, 11, 3 ],
    [ 7, 7, 2, 10 ],
    [ 8, 8, 6, 4 ],
    [ 10, 10, 1, 5 ]
]
> JunNec(r,A);
false [ 7, 7, 2, 10 ]
```

The function JunNec(r,A) checks whether every age 2 point is the sum of two age 1 points. In this case it fails for [7,7,2,10] and [10,10,1,5].

Example 3.6.5. The code shows that $\frac{1}{39}(1,5,8,25)$ has no crepant resolution. First calculate the junior points in the lattice $\mathbb{Z}^4 + \frac{1}{39}(1,5,8,25) \cdot \mathbb{Z}$:

```
> r:=39;
> A:=[1,5,8,25];
> Juniors := FindJuniors(r,A);
> Juniors;
```

```
[
    [ 1, 5, 8, 25 ],
    [ 2, 10, 16, 11 ],
    [ 5, 25, 1, 8 ],
    [ 8, 1, 25, 5 ],
    [ 10, 11, 2, 16 ],
    [ 11, 16, 10, 2 ],
    [ 16, 2, 11, 10 ],
    [ 25, 8, 5, 1 ]
]
```

There are only eight interior lattice points, which is relatively small given the order of the group.

The algorithm computes all basic cones, of which there are 161. It uses this to find a face tiling, before starting to fill the interior of the junior simplex.

```
> CrepantCones := MakeCrepantCones(r,A);
> #CrepantCones;
161
> FacePieces:=FindFaces(Juniors,CrepantCones);
> FacePieces;
[ 132, 149, 161, 63 ]
> CrepantCones[FacePieces];
[
    [
        [ 5/39, 25/39, 1/39, 8/39 ],
        [ 1, 0, 0, 0 ],
        [ 0, 1, 0, 0 ],
        [ 0, 0, 0, 1 ]
    ],
    [
        [ 8/39, 1/39, 25/39, 5/39 ],
        [ 1, 0, 0, 0 ],
        [ 0, 0, 1, 0 ],
        [ 0, 0, 0, 1 ]
    ],
    [
```

```
        [ 25/39, 8/39, 5/39, 1/39 ],
        [ 1, 0, 0, 0 ],
        [ 0, 1, 0, 0 ],
        [ 0, 0, 1, 0 ]
    ],
    [
        [ 1/39, 5/39, 8/39, 25/39 ],
        [ 0, 1, 0, 0 ],
        [ 0, 0, 1, 0 ],
        [ 0, 0, 0, 1 ]
    ]
]
```

Here the face tiling is unique as the a_i are coprime to $r = 39$.

Cones whose cone chains overlap themselves are not permitted in the resolution. In this example only 97 of a possible 161 cones have cone chains which do not overlap.

```
>AdjacencyGraph :=  MakeAdjacencyGraph(CrepantCones);
> ConeChains:=MakeConeChains(CrepantCones,AdjGraph);
> OverlapGraph := MakeOverlapGraph(CrepantCones);
> ConeChainOverlapGraph := MakeConeChainOverlaps(ConeChains,
                                               OverlapGraph);
> AllowedCones := [1..#CrepantCones];
>       for i in [1..#ConeChainOverlapGraph] do
for>    if ConeChainOverlapGraph[i,i] then
for|if>                 Exclude(~AllowedCones,i);
for|if>                 Exclude(~FacePieces,i);
for|if>         end if;
for> end for;
> #AllowedCones;
97
```

The next step in the algorithm is to add in the cone chains of each of the cones in the face tiling. The first cone in the face tiling is cone 132. Its cone chain contains itself and 21 other cones. The other cones of the face tiling (cones 149, 161, 63) are contained in this cone chain, so the only cones which are forced by the face tiling belong to the cone chain of cone 132.

```
> FacePieces[1];
132
> ConeChains[132];
[ 132, 121, 156, 62, 61, 146, 149, 42, 45, 63, 19, 80, 4, 34, 115,
143, 159, 161,120, 130, 142, 147 ]
> #ConeChains[132];
22
> ChosenFaces:=FacePieces;
> ChosenCones:= MakeForced(ConeChains,ChosenFaces);
> #Fo;
22
```

At this stage, there are no unique neighbours at any face of any cone in the set Fo of chosen cones. Thus a search through all faces reveals the possible neighbours at that face.

```
> Choices := MakeChoices(CrepantCones, ChosenCones,AdjacencyGraph,
                          ConeChainOverlapGraph, AllowedCones);
> #Choices;
20
> Choices;
[
    [ 124, 128 ],
    [ 9, 38 ],
    [ 1, 2, 3 ],
    [ 85, 110, 140 ],
    [ 111, 141 ],
    [ 89, 144 ],
    [ 90, 144 ],
    [ 17, 59 ],
    [ 52, 57 ],
    [ 18, 59 ],
    [ 9, 12 ],
    [ 52, 152 ],
    [ 68, 108 ],
    [ 111, 113 ],
    [ 124, 157 ],
```

```
    [ 26, 29 ],
    [ 24, 116, 118 ],
    [ 26, 117 ],
    [ 68, 71 ],
    [ 7, 36, 41 ]
]
```

Cones 124 and 128 are neighbours of cone 130.

```
> CrepantCones[130];
[
    [ 5/39, 25/39, 1/39, 8/39 ],
    [ 25/39, 8/39, 5/39, 1/39 ],
    [ 1, 0, 0, 0 ],
    [ 0, 1, 0, 0 ]
]
> CrepantCones[124];
[
    [ 5/39, 25/39, 1/39, 8/39 ],
    [ 11/39, 16/39, 10/39, 2/39 ],
    [ 25/39, 8/39, 5/39, 1/39 ],
    [ 0, 1, 0, 0 ]
]
> CrepantCones[128];
[
    [ 5/39, 25/39, 1/39, 8/39 ],
    [ 16/39, 2/39, 11/39, 10/39 ],
    [ 25/39, 8/39, 5/39, 1/39 ],
    [ 0, 1, 0, 0 ]
]
```

Exactly one of these must belong to the resolution.

The algorithm runs a tree search on the set of choices, making more if necessary, until it has tried every possible choice. The function returns false, which shows that it has not found a resolution, and the variable resolution is empty.

```
> time boolean,resolution:=HasCrepRes(39,[1,5,8,25]);
Time: 271.750
```

```
> boolean;
false
> resolution;
[]
```

Thus this singularity has no crepant resolution.

Chapter 4

A-Hilb($\mathbb{C}^4$) and crepant resolutions

Unlike in dimension three where the A-Hilbert scheme is a crepant resolution of $\mathbb{C}^3/A$ for A any finite abelian subgroup of $\mathrm{SL}(3, \mathbb{C})$, we see that in dimension four A-Hilb($\mathbb{C}^4$) may be discrepant or even singular. We restrict to looking at the family $\frac{1}{r}(1, 1, a, b)$. Considering only those examples which satisfy condition JunNec 2.2.1 we calculate the A-Hilbert scheme and show that it is at worst a blow-up of a crepant resolution $\mathbb{C}^4/G$.

4.1 Examples

Nakamura [Nak01] proved that the A-Hilbert scheme A-Hilb($\mathbb{C}^3$) gives a crepant resolution of $\mathbb{C}^3/A$. In four dimensions this is not necessarily true, even for examples which satisfy JunNec (Condition 2.2.1). In fact, even if $\mathbb{C}^4/A$ is crepant A-Hilb($\mathbb{C}^4$) maybe discrepant or even singular. We give some examples.

For small values of r the A-Hilbert scheme A-Hilb($\mathbb{C}^4$) and the crepant resolution are the same.

Example 4.1.1. Take $\frac{1}{8}(1, 1, 2, 4)$. This has 3 internal junior points

$$p_1 = (1, 1, 2, 4) \quad p_2 = (2, 2, 4, 0) \quad p_4 = (4, 4, 0, 0)$$

There are exactly 8 basic cones inside the junior simplex and these form the unique crepant resolution.

$$\begin{array}{cccc} e_4e_3e_2p_1 & e_4e_3p_1e_1 & e_4e_2p_1p_4 & e_4p_1p_4e_1 \\ e_3e_2p_1p_2 & e_3p_1p_2e_1 & e_2p_1p_2p_4 & p_1p_2p_4e_1. \end{array}$$

Calculating the affine pieces of the A-Hilbert scheme gives exactly the same cones.

In the following example A-Hilb is discrepant even though there is a crepant resolution.

Example 4.1.2. Consider the lattice $L = \mathbb{Z}^4 + \frac{1}{12}(1,2,3,6)$. This has five internal junior points:

$$p_1 = (1,2,3,6), \quad p_2 = (2,4,6,0), \quad p_4 = (4,8,0,0),$$
$$p_6 = (6,0,6,0), \quad p_8 = (8,4,0,0).$$

There is a crepant resolution given by

$$\begin{array}{cccccc} p_1e_2e_3e_4 & p_2p_1e_2e_3 & p_4p_1e_2e_4 & p_4p_2p_1e_2 & p_6p_1e_3e_4 & p_6p_2p_1e_3 \\ p_8p_4p_1e_4 & p_8p_4p_2p_1 & p_8p_6p_2p_1 & e_1p_6p_1e_4 & e_1p_8p_1e_4 & e_1p_8p_6p_1. \end{array}$$

However A-Hilb consists of 15 nonsingular cones. Nine of these are crepant:

$$\begin{array}{cccccc} e_4e_3e_2p_1 & e_4e_3p_1p_6 & e_4e_2p_1p_4 & e_4p_1p_4p_8 & e_3e_2p_1p_2 & e_3p_1p_2p_6 \\ e_2p_1p_2p_4 & p_1p_2p_4p_8 & p_1p_2p_6p_8, & & & \end{array}$$

and six have discrepancy 1:

$$e_4p_1p_6P_{13} \quad e_4p_1p_8P_{13} \quad e_4p_6e_1P_{13} \quad e_4p_8e_1P_{13} \quad p_1p_6p_8P_{13} \quad p_6p_8e_1P_{13}.$$

Note that the discrepant cones contain the age two point $P_{13} = \frac{1}{13}(13,2,3,6)$. We can see, by comparing the faces of these cones, that they glue together around P_{13}. Cone $P_{13}p_8p_1p_6$ has faces:

$$P_{13}p_8p_1 \quad P_{13}p_8p_6 \quad P_{13}p_1p_6 \quad p_8p_1p_6.$$

The three faces of this cone containing P_{13} belong to one of the five other discrepant cones. Glueing these six cones together gives a new, nonbasic, cone. This is the cone shown in Figure 4.1, with each of the triangles joined to p_1 (going into the page) and e_1 (coming out of the page).

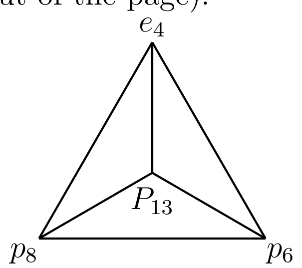

Figure 4.1: Cross section of the cone $p_1e_4p_8p_6e_1$

Let us compare with the crepant resolution.

Crepant resolution	G-Hilb
$p_1e_2e_3e_4$	$p_1e_2e_3e_4$
$p_2p_1e_2e_3$	$p_2p_1e_2e_3$
$p_4p_1e_2e_4$	$p_4p_1e_2e_4$
$p_4p_2p_1e_2$	$p_4p_2p_1e_2$
$p_6p_1e_3e_4$	$p_6p_1e_3e_4$
$p_6p_2p_1e_3$	$p_6p_2p_1e_3$
$p_8p_4p_1e_4$	$p_8p_4p_1e_4$
$p_8p_4p_2p_1$	$p_8p_4p_2p_1$
$p_8p_6p_2p_1$	$p_8p_6p_2p_1$
$e_1p_6p_1e_4$	
$e_1p_8p_1e_4$	
$e_1p_8p_6p_1$	
	$e_4p_1p_6P_{13}$
	$e_4p_1p_8P_{13}$
	$e_4p_6e_1P_{13}$
	$e_4p_8e_1P_{13}$
	$p_1p_6p_8P_{13}$
	$p_6p_8e_1P_{13}$

Contracting the point P_{13} gives cones $p_1e_1p_6p_8, p_1e_1p_8e_4, p_1e_1e_4p_6$, which gives the crepant resolution.

In this example, we have a crepant resolution, but A-Hilb is singular.

Example 4.1.3. Consider $A = \frac{1}{15}(1, 3, 5, 6)$.

There are 5 points in the interior of the junior simplex:

$$\begin{aligned} &p_1 = \tfrac{1}{15}(1,3,5,6), \quad p_3 = \tfrac{1}{15}(3,9,0,3), \qquad p_5 = \tfrac{1}{15}(5,0,10,0), \\ &p_6 = \tfrac{1}{15}(6,3,0,6), \quad p_{10} = \tfrac{1}{15}(10,0,5,0). \end{aligned}$$

A crepant resolution of $\mathbb{C}^4/A$ is

$$\begin{array}{lllll} e_4e_3e_2p_1 & e_4e_3p_1p_5 & e_4e_2p_1p_3 & e_4p_1p_3p_6 & e_4p_1p_5p_{10} \\ e_4p_1p_6e_1 & e_4p_1p_{10}e_1 & e_3e_2p_1p_3 & e_3e_2p_3p_5 & e_3p_1p_3p_5 \\ e_2p_3p_5p_{10} & e_2p_3p_{10}e_1 & p_1p_3p_5p_{10} & p_1p_3p_6e_1 & p_1p_3p_{10}e_1. \end{array}$$

However A-Hilb is singular. A-Hilb consist of 7 crepant cones:

$$\begin{array}{llll} e_4e_3e_2p_1 & e_4e_3p_1p_5 & e_4e_2p_1p_3 & e_4p_1p_3p_6 \\ e_4p_6p_{10}e_1 & e_2p_3p_{10}e_1 & p_3p_6p_{10}e_1, & \end{array}$$

19 discrepant cones:

$$\begin{array}{llll} e_4p_1p_5p_{11} & e_4p_1p_6p_{11} & e_4p_5p_{10}p_{11} & e_4p_6p_{10}p_{11} \\ e_3e_2p_1P_3 & e_3e_2P_3p_5 & e_3p_1P_3p_5 & e_2p_1p_3P_6 \\ e_2p_1P_3P_6 & e_2p_3P_6p_8 & e_2p_3p_8p_{10} & e_2p_5p_8p_{10} \\ p_1p_3p_6p_{11} & p_1p_3P_6p_8 & p_1p_3p_8p_{11} & p_1p_5p_8p_{11} \\ p_3p_6p_{10}p_{11} & p_3p_8p_{10}p_{11} & p_5p_8p_{10}p_{11}, & \end{array}$$

and 2 singular cones whose affine pieces are given by the equations

$$x^3 = \xi t^3, \quad x^2t^3 = \eta z, \quad y = \zeta t^3, \quad z^2 = \lambda x t^4, \quad zt^2 = \mu x^2$$

and

$$x^3 = \xi y, \quad x^2 t = \eta z, \quad yz = \zeta x^2 t, \quad z^2 = \lambda xyt, \quad t^3 = \mu y.$$

These pieces have singularities $\xi\mu = \eta\lambda$ and $\xi\zeta = \eta\lambda$ respectively; these are singularities of the form (3-fold node) $\times \mathbb{A}^1$.

4.2 The family $\frac{1}{r}(1,1,a,b)$

We turn our attention to a smaller family of examples, namely $\frac{1}{r}(1,1,a,b) \subset \mathrm{SL}(4,\mathbb{C})$.

First consider the subfamily given by fixing $a = 7$. The family $\frac{1}{r}(1,1,7,r-9)$ satisfies JunNec whenever r is equal to 0, 7, 9, 14, 16, 21, 23, 27, 28, 30, 34, 37, 41, 48, 55 modulo 63. We will see later that a crepant resolution exists for all these values of r.

Putting these numbers into a table (Table 4.1) based on their decomposition into a sum of multiples of 7 and 9, say $r = 7c + 9d \mod 63$, is quite striking. We see that there is only a crepant resolution (when r is bold) if $c \in \{0,1,2,3,4\}$ and $d \in \{0,1,3\}$.

This is equivalent to the condition that $r = 7s + u = 9t + v$ for some integers s, t, and for $u \in \{0,2,6\}$ and $v \in \{0,1,3,5,7\}$.

For $\frac{1}{r}(1,1,a,b)$ to satisfy JunNec the points $\frac{1}{r}(\alpha,\alpha,1,\beta)$ and $\frac{1}{r}(\gamma,\gamma,\delta,1)$ must have age 1. For this to be the case it is clear that the inverses of a and b modulo r must be less than $\frac{r}{2}$.

For $\frac{1}{r}(1,1,7,r-9)$ if $r = 7s + u = 9t + v$ the inverses c_7 and c_9 of $7 \mod r$ and $9 \mod r$ respectively are given in Table 4.2. It is clear from the table that if $u \in \{1,4,5\}$ or $v \in \{2,4,8\}$ then $\frac{1}{r}(1,1,7,r-9)$ cannot satisfy JunNec.

		d						
		0	1	2	3	4	5	6
	0	**0**	**9**	18	**27**	36	45	54
	1	**7**	**16**	25	**34**	43	52	61
	2	**14**	**23**	32	**41**	50	59	5
	3	**21**	**30**	39	**48**	57	3	12
c	4	**28**	**37**	46	**55**	1	10	19
	5	35	44	53	62	8	17	26
	6	42	51	60	6	15	24	33
	7	49	58	4	13	22	31	40
	8	56	2	11	20	29	38	47

Table 4.1: Values of $r \mod 63$ for which $\frac{1}{r}(1,1,7,r-9)$ satisfies JunNec

u	c_7	v	c_9
1	$6s{+}1$	1	t
2	$3s{+}1$	2	$5t{+}1$
3	$2s{+}1$	4	$7t{+}3$
4	$5s{+}3$	5	$2t{+}1$
5	$4s{+}3$	7	$4t{+}3$
6	$s{+}1$	8	$8t{+}7$

Table 4.2: The inverses of 7 and 9 modulo r

Lemma 4.2.1. *Let h (respectively g) be the highest common factor of a and r (respectively $a+2$ and r). The points $\frac{1}{r}(\alpha,\alpha,h,\beta)$ and $\frac{1}{\mathrm{r}}(\gamma,\gamma,\delta,g)$ are junior if and only if $\beta < \frac{a}{2}-1$ and $\delta < \frac{a+2}{2}-1$.*

Proof. Let $h = \operatorname{hcf}(r,a)$. Write $a = hs$ and $r = ht$. Then there exist integers c and x such that

$$ac = h + xr. \tag{4.1}$$

That is, if $h = 1$, then c is the inverse of a modulo r, and x is the inverse of r modulo a. Dividing (4.1) through by h we get

$$sc = 1 + xt,$$

and dividing this by st gives

$$\frac{c}{t} = \frac{1}{st} + \frac{x}{s}.$$

If the point $\frac{1}{r}(c,c,h,\overline{c(r-a-2)})$ is junior then $r = 2c + h + \overline{c(r-a-2)}$, so we must have $c > \frac{r}{2} - 1$. Thus

$$\frac{h}{2} = \frac{ht}{2t} > \frac{c}{t} = \frac{x}{s} + \frac{1}{st},$$

and we have $x < \frac{a}{2} - 1$ since $t \geq s + 1$. Conversely, if $x < \frac{a}{2} - 1$ then

$$\frac{c}{t} = \frac{x}{s} + \frac{1}{st} < \frac{a}{2s} - \frac{1}{s} = \frac{hs}{2s} - \frac{1}{s} = \frac{r}{2t} - \frac{1}{s},$$

so $c < \frac{r}{2} + \frac{1-t}{s} < \frac{r}{2} - 1$. Then $r > 2c + 2$ so the point $\frac{1}{r}(c, c, h, \overline{c(r-a-2)})$ is junior. Setting $\beta = x$ and $\alpha = c$ gives the result. □

We have proved that for certain values of r the quotient singularity $\frac{1}{r}(1,1,a,b)$ does not satisfy JunNec and so does not have a crepant resolution. Shortly we will prove that in the cases where JunNec is satisfied the A-Hilbert scheme can be contracted to give a crepant resolution. This means that for any value of a we can find a table like Table 4.1 from which we can easily read off the values of r for which a crepant resolution exists.

Our Theorem 4.5.3 gives necessary and sufficient conditions for members of this family to have a crepant resolution. Dais, Haus and Henk [DHH98] prove that a different set of conditions is necessary and sufficient. Their proof is constructive, however their conditions require lengthy calculations. It seems to be a difficult question to determine an exact relation between the two sets of conditions.

4.3 A-Hilbert schemes for $\frac{1}{r}(1,1,a,b)$

We consider resolutions of quotient singularities of the form $\frac{1}{r}(1,1,a,b)$. In these cases all the junior points lie on the plane through e_3, e_4 and the midpoint of the axis $A = e_1e_2$. We shall denote this point $(\frac{r}{2}, \frac{r}{2}, 0, 0)$ by A'. Note that A' is a lattice point if and only if both a and b are even.

The idea is to find a triangulation of the median triangle, e_3e_4A', into basic triangles in a similar way to the Craw-Reid algorithm [CR02]. Once a triangulation of e_3e_4A' has been chosen, for every basic triangle $p_1p_2p_3$ which does not have A' as a vertex we form the two tetrahedra with vertices p_1, p_2, p_3, e_1 and p_1, p_2, p_3, e_2. If A' is a lattice point we do the same. Otherwise, we replace A' with both the vertices e_1 and e_2. Thus we obtain a tiling of the junior simplex, Δ, into basic tetrahedra, and this gives a crepant resolution.

4.4 The case a and b even

If both a and b are even, then r is even and the point A' is a junior point. Say $r = 2r', a = 2a'$ and $b = 2b'$, then the original Craw-Reid algorithm for $\frac{1}{r'}(1, a', b')$ gives a triangulation of the simplex e_3e_4A', and G-Hilb$(\mathbb{C}^4)$ is the crepant resolution corresponding to this.

Example 4.4.1. $\frac{1}{26}(1, 1, 4, 20)$. We take $r' = 13, a' = 2$ and $b' = 10$. We found the crepant resolution of $\frac{1}{13}(1, 2, 10)$ in Example 1.4.5. Each triangle of Figure 4.2

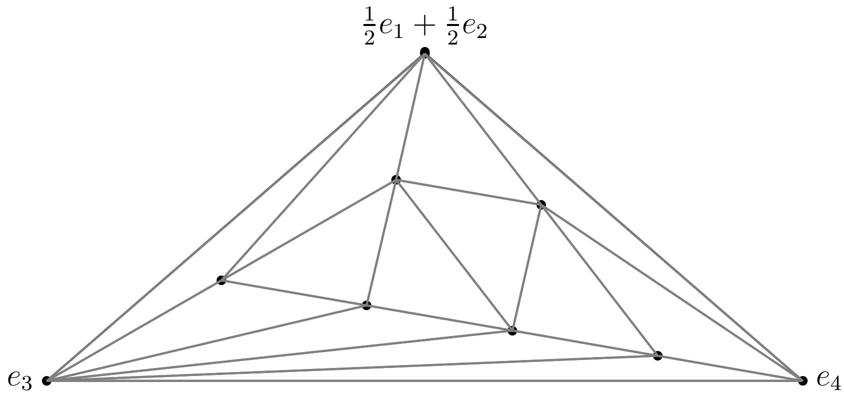

Figure 4.2: A-Hilb $\mathbb{C}^4$ for $\frac{1}{26}(1, 1, 4, 20)$

forms a tetrahedron with the addition of the vertex e_1, similarly for the vertex e_2. The G-Hilbert scheme consists of these 26 affine pieces.

We saw that the regular triangle $f_{3,1} = \frac{1}{13}(1, 2, -3), f_{1,2} = \frac{1}{13}(-5, 3, 2), f_{2,1} = \frac{1}{13}(4, -5, 1)$ of side 1 has dual basis

$$\xi = x^2/y, \quad \eta = y^2/z^3, \quad \zeta = z^4/x$$

which gives equations $x^2 = \xi y, y^2 = \eta z^3, z^4 = \zeta x$.

The corresponding triangle $f_{4,1}, f_{A,2}, f_{3,1}$ for $\frac{1}{26}(1, 1, 4, 20)$ gives a tetrahedron when the vertex e_2 is added. This tetrahedron has dual basis

$$\xi = x^4/z, \quad \eta = z^2/t^3, \quad \zeta = t^4/x^2, \quad \theta = x/y.$$

These give equations $x^4 = \xi z, z^2 = \eta t^3, t^4 = \zeta x^2, y = \theta x$, which lead to the equations $z^2t = \lambda x^2, \quad x^2t^4 = \mu z, \quad x^4z = \nu t^3$ since x^2zt is an invariant monomial.

The corresponding tetrahedron with vertex e_1 has dual basis

$$\xi = y^4/z, \quad \eta = z^2/t^3, \quad \zeta = t^4/y^2, \quad \theta = y/x.$$

Similar equations arise with x and y interchanged.

For the general case, consider the group generated by $\frac{1}{2r}(1,1,2a,2b)$. All junior points lie on the plane $(x-y)$ through e_3, e_4, A'. Thus we may consider the group $\frac{1}{r}(1,a,b) \in \mathrm{SL}(3,\mathbb{C})$. The Craw-Reid algorithm gives a subdivision of the triangle e_3e_4A' into regular triangles with dual bases which give generators for the A-clusters.

Each of the Craw-Reid regular triangles corresponds to two regular tetrahedra in Δ: one is given by including the vertex e_1, the second by including the vertex e_2. When we convert from the lattice $\mathbb{Z}^3+\frac{1}{r}(1,a,b)\cdot\mathbb{Z}$ to the lattice $\mathbb{Z}^4+\frac{1}{2r}(1,1,2a,2b)\cdot\mathbb{Z}$, the z and t coordinates are doubled. Thus the exponents of x and y in the dual vectors must be doubled. The dual basis of the regular triangles consists of only three elements. We must add the element $\theta = y/x$, or its inverse, to the dual basis of each regular triangle to make the dual basis of a regular tetrahedron. There are two types of tetrahedra; those with vertex e_1, which we will refer to as being "above" the triangle $A'e_3e_4$ and with the equation $x = \theta y$, and those with vertex e_2, which we will refer to as "below", with the equation $y = \theta x$. These equations mean that x (respectively y) will not appear in any of the other equations of that tetrahedron. Thus Nakamura's Theorem becomes

Theorem 4.4.2. *(I). For every finite diagonal subgroup* $A = \frac{1}{2r}(1,1,2\alpha,2\beta) \subset \mathrm{SL}(4,\mathbb{C})$ *and every A-cluster Z generators, of the ideal $\mathcal{I}_Z$, can be chosen as a system of 8 equations. In the "below" case:*

$$\begin{aligned} x^{l+1} &= \xi z^b t^f, & z^{b+1}t^{f+1} &= \lambda x^{l-1}, \\ z^{m+1} &= \eta x^d t^c, & x^{d+2}t^{c+1} &= \mu z^m, \\ t^{n+1} &= \zeta x^a z^e, & x^{a+2}z^{e+1} &= \nu t^n, \\ y &= \theta x, & xyzt &= \pi. \end{aligned} \tag{4.2}$$

Here $a,b,c,d,e,f,l,m,n \geq 0$ are integers, and $\xi,\eta,\zeta,\lambda,\mu,\nu,\pi \in \mathbb{C}$ are constants satisfying

$$\lambda\xi\theta = \mu\eta\theta = \nu\zeta\theta = \pi.$$

(II). Moreover, exactly one of the following cases holds:

$$\text{“Up”}\quad \begin{cases} \lambda = \eta\zeta, \mu = \zeta\xi, \nu = \xi\eta, \pi = \xi\eta\zeta\theta \\ l = a + d, m = b + e, n = c + f; \textit{or} \end{cases}$$

$$\text{“Down”}\quad \begin{cases} \xi = \mu\nu, \quad \eta = \nu\lambda, \quad \zeta = \lambda\mu, \quad \pi = \lambda\mu\nu\theta, \\ l = a + d + 1, \quad m = b + e + 1, \quad n = c + f + 1. \end{cases}$$

The “above” case is given by interchanging x and y in every equation.

4.5 Calculating continued fractions

From now on we will assume that at least one of a or b is odd. We follow a Craw-Reid type algorithm [CR02] computing the continued fractions $\frac{r}{a}$ and $\frac{r}{b}$ to give lines out of the vertices e_4 and e_3 respectively. At A' we do the following similar calculation.

Let $h = \mathrm{hcf}(r, a)$. If $h = 1$ there is a unique $1 \leq c < r$ such that

$$h = ac + rx.$$

However, for $h > 1$, there are several possible values for c. The plane $x_3 = h$ is parallel to the face Ae_4. We choose the c for which $\overline{r - 2c - h}$ is smallest, thus the point $p_c = \frac{1}{r}(c, c, h, \overline{r - 2c - h})$ is the closest point on this plane to the point A'. We take the Hirzebruch-Jung continued fraction

$$\frac{r}{h(r - 2c - h)} = [b_1, b_2, \ldots, b_k], \tag{4.3}$$

and run the continued fraction algorithm to compute the planes out of A as lines out of A'.

If p_c has age 2 then JunNec fails immediately. If $h = 1$, it is clear that the point $\frac{1}{\mathrm{r}}(c, c, 1, 2r - 2c - 1)$ cannot be expressed as the sum of two junior points. For $h > 1$, suppose there are two junior points p_i, p_j such that $p_c = p_i + p_j$. Then

$$\frac{1}{\mathrm{r}}(c, c, h, 2r - 2c - h) = \frac{1}{\mathrm{r}}(i, i, h, r - 2i - h) + \frac{1}{\mathrm{r}}(j, j, 0, r - 2j).$$

However we chose c so that $2r - 2c - h = \overline{r - 2c - h}$ has smallest value, so $r - 2i - h > 2r - 2c - h$, consequently p_c cannot be expressed as the sum of two junior points.

Condition 4.5.1. The point p_c is junior and all entries b_i of the continued fraction (4.3) are even.

Lemma 4.5.2. *The continued fraction algorithm returns only junior lattice points if and only if Condition 4.5.1 is satisfied.*

Proof. The continued fraction algorithm produces points in the plane Π_1 through A', e_4 and p_c, so it can only produce junior points if Π_1 coincides with the plane Π_2 through A', e_3 and e_4. It is clear that Π_1 and Π_2 only coincide if p_c lies in Π_2, that is, if p_c is junior.

Suppose p_c is the closest point to the face $e_4 A$ and $[b_1, \ldots, b_k]$ is the Hirzebruch-Jung continued fraction expansion of

$$\frac{r}{h(r-2c-h)} = [b_1, b_2, \ldots, b_k].$$

Then the continued fraction says that the next lattice point is given by the vector out of A'

$$\begin{aligned} &b_1\left(c - \frac{r}{2}, c - \frac{r}{2}, h, r-2c-h\right) - \left(-\frac{r}{2}, -\frac{r}{2}, 0, r\right) \\ &\quad = \left(b_1 c - \frac{(b_1-1)r}{2}, b_1 c - \frac{(b_1-1)r}{2}, b_1 h, (b_1-1)r - b_1(2c+h)\right). \end{aligned}$$

This gives the point

$$p = \left(b_1 c - \frac{(b_1-2)r}{2}, b_1 c - \frac{(b_1-2)r}{2}, b_1 h, (b_1-1)r - b_1(2c+h)\right).$$

It is clear that if b_1 is even then p is a lattice point. For the converse, if b_i were odd then the point p being a lattice point would imply that $A' = (\frac{r}{2}, \frac{r}{2}, 0, 0)$ were also a lattice point, since

$$\begin{aligned} p &= \left(b_1 c - \frac{(b_1-2)r}{2}, b_1 c - \frac{(b_1-2)r}{2}, b_1 h, (b_1-1)r - b_1(2c+h)\right) \\ &= (b_1 c, b_1 c, b_1 h, (b_1-1)r - b_1(2c+h)) - \left(\frac{(b_1-1)}{2}r, \frac{(b_1-1)}{2}r, 0, 0\right) \\ &\quad - \left(\frac{r}{2}, \frac{r}{2}, 0, 0\right). \end{aligned}$$

However, A' is only a lattice point if a, b are both even, thus b_1 must be even. $\square$

Theorem 4.5.3 (Main Theorem)**.** *There exists a crepant resolution if and only if Condition 4.5.1 is satisfied.*

Proof that Condition 4.5.1 is necessary. We have already showed that for a crepant resolution to exist p_c must be junior. If at least one of the b_i is not even then in Lemma 4.5.2 we proved that the continued fraction algorithm returns a point which is not junior. The first step in the continued fraction algorithm takes the lattice points

$$e_4 = (0, 0, 0, r) \text{ and } p_c = (c, c, h, r - 2c - h),$$

and chooses the lattice point with the smallest third coordinate greater than h and whose fourth coordinate is less than $r - 2c - h$. The algorithm gives us a chain of points

$$\ldots,\ p_1 = (x_1, x_1, y_1, z_1),\ p_2 = (x_2, x_2, y_2, z_2),\ p_3 = (x_3, x_3, y_3, z_3),\ \ldots$$

where the y_i are strictly increasing and the z_i are strictly decreasing. If p_2 is not junior then it cannot be the sum of junior points since any lattice point whose third coordinate is between y_1 and y_2 will have fourth coordinate greater than z_2, and similarly for any lattice point whose fourth coordinate is between z_2 and z_3. □

The sufficiency of Condition 4.5.1 will be proved at the end of section 4.10. The idea is as follows.

If Condition 4.5.1 is satisfied then an argument based on [CR02] gives an algorithm to compute the A-Hilbert scheme. Contraction of the divisors in the A-Hilbert scheme gives a crepant resolution. If either of the conditions fail then the same algorithm provides a lattice point of age 2 that is not the sum of two juniors, showing that no crepant resolution can exist.

From now on we will denote the entries of the continued fraction at e_i as $[b_{i,1}, \ldots, b_{i,k_i}]$ for $i = 3, 4$. We denote by $[b_{A,1}, \ldots, b_{A,k_A}]$ the continued fraction at A'. Similarly $f_{i,j}$ denotes a vector out of the vertex e_i if $i = 3, 4$ or A' if $i = A$. We consider the vertices A', e_3, e_4 to be in a cycle so that $f_{i-1,j}$ is a vector out of the previous vertex in the cycle. The number $b_{i,j}$ is called the *strength* of $f_{i,j}$.

A *primitive vector* v is a vector such that if v is the vector between two lattice points then there are no other lattice points on v. We will call w a *half-primitive* vector if $2w$ is a primitive vector.

The vectors $v_1, v_2, v_3 \in \mathbb{Z}^2$ form a *regular triple* if any two of them form a basis of $\mathbb{Z}^2$ and such that $\pm v_1 \pm v_2 \pm v_3 = 0$. We will call a set of vectors v_1, v_2, v_3 a *half-regular triple* if one of the following holds:

1. $2v_1, v_2, v_3$ are primitive vectors, any two of which form a basis of $\mathbb{Z}^2$ and such that $\pm 2v_1 \pm v_2 \pm v_3 = 0$

2. $2v_1, 2v_2, v_3$ are primitive vectors, any two of which form a basis of $\mathbb{Z}^2$ and such that $2v_1 \pm 2v_2 \pm v_3 = 0$

3. $2v_1, 2v_2, v_3$ are primitive vectors, any two of which form a basis of $\mathbb{Z}^2$ and such that $2v_1 \pm 2v_2 \pm 2v_3 = 0$.

From now on we will use the term half-regular triple to include the possibility that the triple is regular, unless otherwise stated.

A triangle $T \subset R^2_\Delta$ is a *lattice triangle* if the vertices of T lie in $\mathbb{Z}^2$. We say that T is a *half-lattice triangle* if the vertices of T lie in $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2}) \cdot \mathbb{Z}$. The triangle T is *regular* if each of its sides is a line L_{ij} extending some $[e_i, f_{i,j}]$ and the three primitive vectors v_1, v_2, v_3 pointing along its sides form a regular triple. We say T is *half-regular* if v_1, v_2, v_3 form a half-regular triple.

Example 4.5.4. We first calculate the continued fractions at each vertex A', e_3, e_4 for the example $\frac{1}{50}(1,1,5,43)$. At e_3 we have $\frac{50}{43} = [2,2,2,2,2,2,8]$, and

$$2f_{3,1} = 2 \cdot \frac{1}{50}(1,1,-45,43) = \frac{1}{50}(0,0,-50,50) + \frac{1}{50}(2,2,-40,33) = f_{3,0} + f_{3,2}.$$

This gives us vectors from e_3 to the points $p_1, p_2, \ldots, p_7$. We also see that

$$8 \cdot \frac{1}{50}(7,7,-15,1) - \frac{1}{50}(6,6,-20,8) = \frac{1}{50}(50,50,-100,0)$$

which is the vector from e_3 through A' to the first lattice point on this line.

At e_4 we have $\frac{50}{5} = 10$, so the only line out of e_4 passes through p_1:

$$10 \cdot \frac{1}{50}(1,1,5,-7) - \frac{1}{50}(0,0,50,-50) = \frac{1}{50}(10,10,0,-20),$$

where $\frac{1}{50}(10,10,0,30)$ is the closest point to e_4 on the line e_4A'.

The calculation at A' is essentially the same, although we must first compute the correct continued fraction. Let $h = \mathrm{hcf}(50,5) = 5$. The possible values of c such that $5 = 5c \mod 50$ are 1, 11, 21, 31, and 41. The point $p_{21} = \frac{1}{r}(21,21,5,3)$ has smallest fourth coefficient, and as such is the closest point to the face Ae_4, so we take $c = 21$. We calculate the continued fraction

$$\frac{r}{h(r-2c-h)} = \frac{50}{15} = [4,2,2].$$

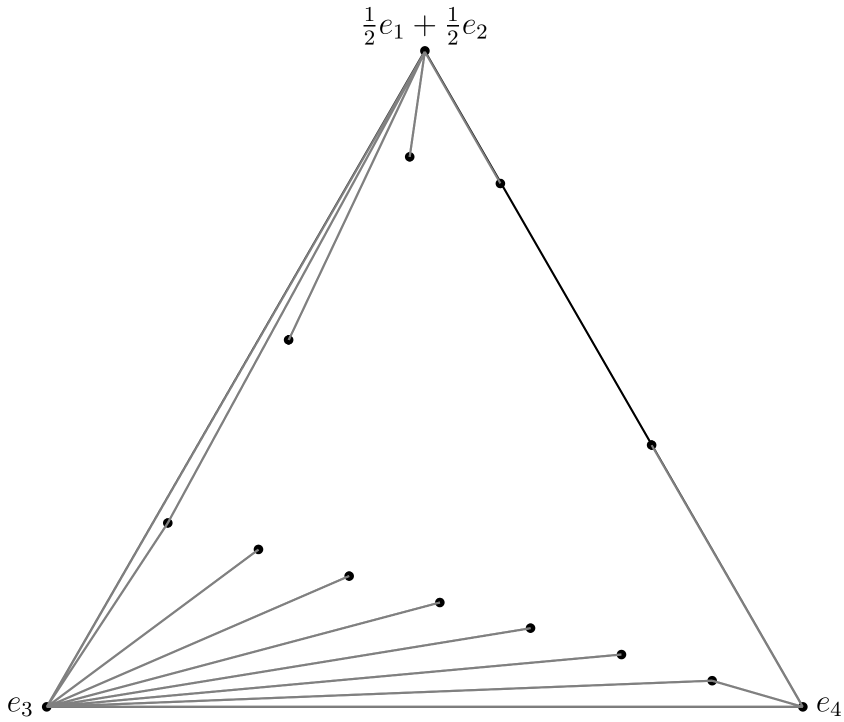

Figure 4.3: Lines out of the vertices for $\frac{1}{50}(1,1,5,43)$

Running the continued fraction algorithm with $f_{A,1} = A' - p_c$ we obtain

$$\begin{aligned} 4f_{A,1} - f_{1,0} = 4 \cdot \frac{1}{50}(-4,-4,5,3) - \frac{1}{50}(-5,-5,0,10) &= \frac{1}{50}(-11,-11,20,2) = f_{1,2}, \\ 2 \cdot \frac{1}{50}(-11,-11,20,2) - \frac{1}{50}(-4,-4,5,3) &= \frac{1}{50}(-18,-18,35,1) = f_{1,3}, \\ 2 \cdot \frac{1}{50}(-18,-18,35,1) - \frac{1}{50}(-11,-11,20,2) &= \frac{1}{50}(-25,-25,50,0) = f_{1,4}, \end{aligned}$$

which are the vectors from A' to the points p_7, p_{14} and p_{21}, as illustrated in Figure 4.3.

Example 4.5.5. The singularity $\frac{1}{31}(1,1,3,26)$ does not have a crepant resolution. At A' we have $h = \mathrm{hcf}(31,3) = 1$ and $c = 21$. The point $p_c = \frac{1}{31}(21,21,1,19)$ has age 2 and the continued fraction $\frac{31}{19} = [2,3,4,2]$ has an entry which is not even. Thus the continued fraction algorithm yields a point which does not belong to the lattice:

$$\begin{aligned} 2 \cdot \left(\frac{-29}{2}, \frac{-29}{2}, 3, 26\right) - \left(\frac{-31}{2}, \frac{-31}{2}, 0, 31\right) &= \left(\frac{-27}{2}, \frac{-27}{2}, 6, 21\right) \\ 3 \cdot \left(\frac{-27}{2}, \frac{-27}{2}, 6, 21\right) - \left(\frac{-29}{2}, \frac{-29}{2}, 3, 26\right) &= \left(\frac{-52}{2}, \frac{-52}{2}, 15, 37\right) \end{aligned}$$

but $(\frac{-21}{2}, \frac{-21}{2}, 15, 37)$ is not a lattice point. Figure 4.4 illustrates this situation; a

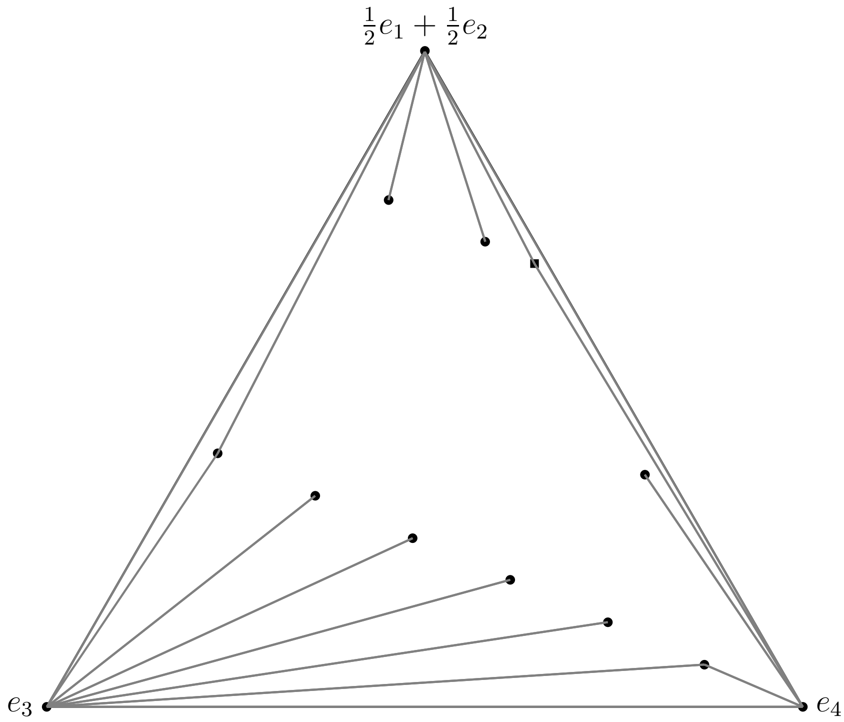

Figure 4.4: Lines out of the vertices for $\frac{1}{31}(1, 1, 3, 26)$

square denotes the point where the ray through p_{21} hits this plane. Since p_{21} has age 2, but is not the sum of two junior points it is clear that no crepant resolution can exist.

4.6 Concatenation of continued fractions

Given a continued fraction $[b_1, \ldots, b_n]$ there exist vectors f_i for $0 \leq i \leq n+1$ satisfying relations

$$b_i f_i = f_{i-1} + f_{i+1}.$$

Figure 4.3 shows the result of running the continued fraction algorithm at each of the vertices A', e_3, e_4 for the example $\frac{1}{50}(1, 1, 5, 43)$. The edges of the triangle also satisfy similar relations:

$$c_j f_{j,0} = f_{j,1} + f_{j-1,k}.$$

This is because, as in [CR02], placing the Newton polygons at A', e_3, e_4 and their inverses at a common vertex gives a basic subdivision of the plane.

In our case $f_{3,0} = (r, r, -2r, 0)$ and $f_{A,k_A+1} = (-\frac{r}{2}, -\frac{r}{2}, r, 0)$ so $f_{3,0} = -2f_{1,k_A+1}$.

We choose c_j to be the integers such that

$$\begin{aligned} c_A f_{A,0} &= f_{A,1} - f_{4,k_4} \\ c_3 f_{A,k_A+1} &= f_{A,k_A} - f_{3,1} \\ c_4 f_{4,0} &= f_{4,1} - f_{3,k_3}. \end{aligned}$$

If $c_j > 1$ then we call this side a *long side.*

As in Craw-Reid we concatenate the continued fractions at the vertices. We must first check whether we have a long side.

Example 4.6.1. There is a long side for $\frac{1}{42}(1, 1, 5, 35)$ since

$$f_{A,2} = \frac{1}{42}(-4, -4, 1, 7), \quad f_{A,3} = \frac{1}{42}(-3, -3, 6, 0), \quad f_{3,1} = \frac{1}{42}(5, 5, -17, 1),$$

satisfy $3f_{A,3} = f_{A,2} - f_{3,1}$.

Note that long sides can only happen in examples in which a and $a + 2$ are not coprime to r.

Lemma 4.6.2. *There is at most one long side.*

Our vectors $f_{A,i}$ are half-lattice vectors. If we take the corresponding lattice vectors $2f_{A,i}$ in the basic subdivision of the upper half-space then the argument of [CR02] holds.

Lemma 4.6.3. *If a and $r - a - 2$ are not both even then $c_4 = 1$.*

Proof. If a and $r - a - 2$ are not both even then there is no $c < r$ such that the vector $(0, 0, c, -c)$ is primitive. We must have $f_{4,0} = f_{4,1} - f_{3,k_3}$. □

Concatenating the continued fractions gives

$$[c_A, b_{A,1}, \ldots, b_{A,k_A}, c_2, b_{3,1}, \ldots, b_{3,k_3}, 1, b_{4,1}, \ldots, b_{4,k_4}]. \tag{4.4}$$

The entries of (4.4) correspond to expressions

$$bv_2 = v_1 + v_3. \tag{4.5}$$

Thus a 1 corresponds to a half-regular triple $v_2 = v_1 + v_3$, which allows us to eliminate v_2 from the expressions of the form (4.5), with v_1, v_3 the vectors corresponding to the entries on either side of the 1. In the 3-dimensional case a 1 could be eliminated by subtracting 1 from each of its neighbours. Here we must be more careful. We have

$$c_3 f_{A,k_A+1} = f_{A,k_1} + f_{3,1},$$

but $f_{A,k_A+1} = (-\frac{r}{2}, -\frac{r}{2}, r, 0)$ is not a primitive vector out of e_3 since A' is not a lattice point. So $f_{3,0} = -2f_{A,k_A+1}$. Hence if $c_3 = 1$ we may contract this 1, but we must subtract 2 from $b_{3,1}$ and 1 from b_{A,k_A}:

$$\begin{aligned} b_{A,k_A} f_{A,k_A} &= f_{A,k_A+1} + f_{A,k_A-1} \\ &= f_{A,k} + f_{3,1} + f_{A,k_A-1}, \end{aligned}$$

so

$$(b_{A,k_A} - 1) f_{A,k_A} = f_{3,1} + f_{A,k_A-1},$$

and

$$\begin{aligned} b_{3,1} f_{3,1} &= 2f_{A,k_A+1} + f_{3,2} \\ &= 2f_{A,k_A} + 2f_{3,1} + f_{3,2}, \end{aligned}$$

so

$$(b_{3,1} - 2) f_{3,1} = 2f_{A,k_A} + f_{3,2}.$$

This factor of 2 must be followed through the calculation.

Lemma 4.6.4. *Let $c_A = 1$ be the strength of the line $f_{A,0}$ out of A'. The contraction of a 1 leads to a chain of contractions. Every contraction in the chain of contractions resulting from the contraction of c_A leads to either:*

$$a, 1, b \to a-1, b-2$$

or

$$a, 1, b \to a-2, b-1.$$

Proof. Suppose we have

$$f_{A,0} = f_{A,1} + f_{4,k}$$

$$b_{A,1} f_{A,1} = f_{A,0} + f_{A,2} \tag{4.6}$$

$$b_{4,k} f_{4,k} = 2f_{A,0} + f_{4,k-1}. \tag{4.7}$$

Then contracting the 1 corresponding to $f_{A,0}$ is equivalent to eliminating $f_{A,0}$ in (4.6) and (4.7). This gives

$$\begin{aligned}(b_{A,1}-1)f_{A,1} &= f_{4,k}+f_{A,2}\\ (b_{4,k}-2)f_{4,k} &= 2f_{A,1}+f_{4,k-1}.\end{aligned}$$

Another contraction can be made if $b_{A,1}-1=1$ and $b_{A,2}>1$, or $b_{4,k}-2=1$ and $b_{4,k-1}>1$. In the first case we get

$$(b_{4,k}-4)f_{4,k}=2f_{A,2}+f_{4,k-1} \quad (b_{A,2}-1)f_{A,2}=f_{4,k}+f_{A,3},$$

and in the second

$$(b_{4,k-1}-1)f_{4,k}=2f_{A,1}+f_{4,k-2} \quad (b_{A,1}-3)f_{A,1}=f_{4,k-1}+f_{A,2}.$$

Thus either

$$a,1,b\to a-1,b-2$$

or

$$a,1,b\to a-2,b-1.$$

□

Example 4.6.5 (Simple Example)**.** $\frac{1}{23}(1,1,3,18)$. The three continued fractions are

$$\begin{aligned}\frac{23}{23-18}&=[3,3,2,2,2] && \text{at } e_3\\ \frac{23}{3}&=[8,3] && \text{at } e_4\\ \frac{23}{6}&=[4,6] && \text{at } A'.\end{aligned}$$

There is no long side because 3 and 18 are coprime to 23 so the concatenation of these continued fractions is

$$[1,4,6,1,3,3,2,2,2,1,8,3]'$$

The contraction of the third 1, works exactly as in the Craw-Reid case: $a,1,b\to a-1,b-1$. Contraction of a 1 eliminates the vector marked with the 1, and so corresponds to deleting a regular triangle.

$$\begin{array}{lll} \text{Step a:} & f_{3,6} = f_{3,5} - f_{4,1}: & \to [1,4,6,1,3,3,2,2,1,7,3] \\ \text{Step b:} & f_{3,5} = f_{3,4} - f_{4,1}: & \to [1,4,6,1,3,3,2,1,6,3] \\ \text{Step c:} & f_{3,4} = f_{3,3} - f_{4,1}: & \to [1,4,6,1,3,3,1,5,3] \\ \text{Step d:} & f_{3,3} = f_{3,2} - f_{4,1}: & \to [1,4,6,1,3,2,4,3] \end{array}$$

Contractions of the other two 1s are more complicated because $f_{3,0} = -2f_{A,3}$ and $f_{4,3} = -2f_{A,0}$. Since $3f_{3,1} = f_{3,0} - f_{3,2} = 2f_{A,3} - f_{3,2}$ and $6f_{A,2} = f_{A,3} - f_{A,1}$, contracting the first 1 corresponds to subtracting 2 from the strength of $f_{3,1}$ and subtracting 1 from the strength of $f_{A,2}$:

$$\text{Step e:} \quad f_{A,3} = f_{A,2} - f_{3,1}: \quad \to [1,4,5,1,3,2,2,2,1,8,3]$$

We now have $f_{3,2} = 2f_{A,3} - f_{3,1}$ and this factor of 2 on $f_{A,3}$ will appear in calculations involving $f_{3,2}$ and the results of such calculations.

$$\text{Step f:} \quad f_{3,1} = 2f_{A,2} - f_{3,2}: \quad \to [1,4,3,2,2,2,2,1,8,3]$$

The calculations involving $f_{A,0}$ are similar:

$$\begin{array}{lll} \text{Step g:} & f_{A,0} = f_{A,1} - f_{4,2}: & \to [3,6,1,3,3,2,2,2,1,8,1] \\ \text{Step h:} & f_{4,2} = 2f_{A,1} - f_{4,1}: & \to [1,6,1,3,3,2,2,2,1,7] \\ \text{Step i:} & f_{A,1} = -f_{4,1} + f_{A,2}: & \to [5,1,3,3,2,2,2,1,5] \end{array}$$

Carrying out these steps in this order gives $[2,1,1]$ which corresponds to the half-regular triple $f_{4,1} = f_{A,2} - f_{A,1}$. This is not unique, but permuting the order of the steps always leads to $[2,1,1]$. If the 2 is attached to a vector out of A' we get a triple of the form $2v_2 = v_1 + v_3$, but if the 2 is attached to a vector out of e_3 or e_4 we get a triple $2v_2 = 2v_1 + 2v_3$, with both v_1 and v_3 vectors out of A'. We also can end at $2f_{4,2} = 2f_{A,1} - 2f_{A,0}$, $2f_{A,1} = f_{4,1} - f_{4,2}$, $2f_{3,1} = 2f_{A,3} + 2f_{A,2}$ and $2f_{A,2} = f_{3,2} - f_{3,1}$. It is not possible to get to any of the regular triples $f_{3,j} = f_{3,j-1} - f_{4,1}$ for $1 \leq j \leq 4$.

Example 4.6.6 (Example with all half-regular triangles)**.** Consider the quotient singularity $\frac{1}{57}(1,1,5,50)$. The highest common factor of 57 and 5 is 1, so $c = 23$

and $r - 2c - h = 10$. The continued fractions are

$$\frac{57}{10} = [6,4,2,2] \quad \text{at } A'$$
$$\frac{57}{7} = [8,2,2,2,2,2,2,2] \quad \text{at } e_3$$
$$\frac{57}{5} = [12,2,3] \quad \text{at } e_4.$$

There are no long sides as the example is coprime, so the concatenation of continued fractions is

$$[6,4,2,2,1,8,2,2,2,2,2,2,2,1,12,2,3,1].$$

Contraction of the 1 on $f_{3,9}$ is exactly as in the Craw-Reid algorithm: $a,1,b \to a-1,b-1$. The contraction eliminates the vector marked with the 1, which corresponds to deleting a regular triangle.

$$\begin{array}{lll}
\text{Step a:} & f_{3,9} = f_{3,8} - f_{4,1}: & \to [6,4,2,2,1,8,2,2,2,2,2,2,1,11,2,3,1] \\
\text{Step b:} & f_{3,8} = f_{3,7} - f_{4,1}: & \to [6,4,2,2,1,8,2,2,2,2,2,1,10,2,3,1] \\
\text{Step c:} & f_{3,7} = f_{3,6} - f_{4,1}: & \to [6,4,2,2,1,8,2,2,2,2,1,9,2,3,1] \\
\text{Step d:} & f_{3,6} = f_{3,5} - f_{4,1}: & \to [6,4,2,2,1,8,2,2,2,1,8,2,3,1] \\
\text{Step e:} & f_{3,5} = f_{3,4} - f_{4,1}: & \to [6,4,2,2,1,8,2,2,1,7,2,3,1] \\
\text{Step f:} & f_{3,4} = f_{3,3} - f_{4,1}: & \to [6,4,2,2,1,8,2,1,6,2,3,1] \\
\text{Step g:} & f_{3,3} = f_{3,2} - f_{4,1}: & \to [6,4,2,2,1,8,1,5,2,3,1] \\
\text{Step h:} & f_{3,2} = f_{3,1} - f_{4,1}: & \to [6,4,2,2,1,7,4,2,3,1].
\end{array}$$

Contractions of the other two 1s are more complicated because $f_{3,0} = 2f_{A,5}$ and $f_{4,4} = 2f_{A,0}$. Since $3f_{4,3} = f_{4,2} + f_{4,4} = f_{4,2} - 2f_{A,0}$ and $6f_{A,1} = f_{A,2} + f_{A,0}$, contracting the third 1 corresponds to subtracting 2 from the strength of $f_{4,3}$ and subtracting 1 from the strength of $f_{A,1}$:

$$\begin{array}{lll}
\text{Step i:} & f_{A,0} = f_{A,1} - f_{4,3}: & \to [5,4,2,2,1,8,2,2,2,2,2,2,2,1,12,2,1] \\
\text{Step j:} & f_{4,3} = f_{4,2} - 2f_{A,1}: & \to [3,4,2,2,1,8,2,2,2,2,2,2,2,1,12,1] \\
\text{Step k:} & f_{4,2} = f_{4,1} - 2f_{A,1}: & \to [1,4,2,2,1,8,2,2,2,2,2,2,2,1,11] \\
\text{Step l:} & f_{A,1} = f_{A,2} - f_{4,1}: & \to [3,2,2,1,8,2,2,2,2,2,2,2,1,9].
\end{array}$$

Similarly, contracting the first 1 corresponds to subtracting 2 from the strength

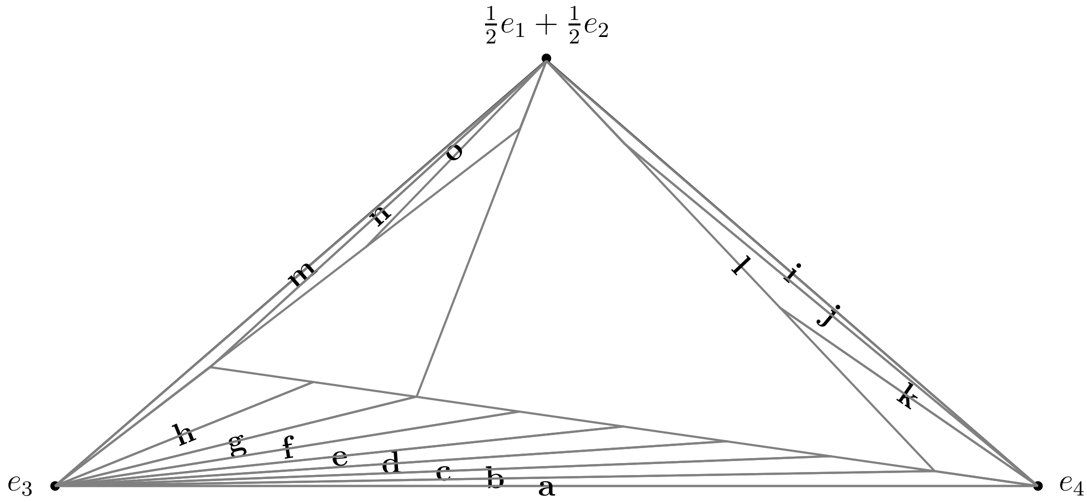

Figure 4.5: Deleting half-regular triangles of $\frac{1}{57}(1, 1, 5, 50)$

of $f_{3,1}$ and subtracting 1 from the strength of f_{14}:

Step m: $f_{A,5} = f_{A,4} - f_{3,1}$: $\rightarrow [6, 4, 2, 1, 6, 2, 2, 2, 2, 2, 2, 2, 2, 1, 12, 2, 3, 1]$
Step n: $f_{A,4} = f_{A,3} - f_{3,1}$: $\rightarrow [6, 4, 1, 4, 2, 2, 2, 2, 2, 2, 2, 1, 12, 2, 3, 1]$
Step o: $f_{A,3} = f_{A,2} - f_{3,1}$: $\rightarrow [6, 3, 2, 2, 2, 2, 2, 2, 2, 2, 1, 12, 2, 3, 1]$.

Contracting in this order leaves us with $[2, 1, 1]$ corresponding to the half-regular triple $2f_{A,2} = -f_{3,1} - f_{4,1}$. This is not unique, however we are always left with $[2, 1, 1]$. Permuting the order of contractions leads to different half-regular triples, for example performing the contractions in this order apart from doing Step l after Step o leads to the triple $f_{4,1} = 2f_{A,2} - 2f_{A,1}$. It is not possible to permute the order to end with a regular triple $f_{3,s} = f_{3,s-1} - f_{4,1}$, for $1 \leq s \leq 8$.

Figure 4.5 shows how Steps a – o delete half-regular triangles.

4.7 Regular triples

In our situation regular triples arise in a slightly different way because vectors out of A' are not *lattice vectors* i.e. they aren't a vector between any two points of the lattice — we must take twice them.

We get the following *half-regular triples*:

1. $\pm v_1 \pm v_2 \pm v_3 = 0$. This happens if either

   (a) v_1 is a vector out of e_i, v_2, v_3 are vectors out of A';

   (b) v_1, v_2, v_3 are vectors out of e_3 or e_4

2. $\pm 2v_1 \pm v_2 \pm v_3 = 0$, with v_1 a vector out of A' and v_2, v_3 vectors out of e_3 or e_4.

3. $\pm 2v_1 \pm 2v_2 \pm 2v_3 = 0$, with v_1 a vector out of A', v_2 a vector along Ae_i and v_3 a vector out of e_j.

Recall that a triangles is half-regular if each of its sides is a line L_{ij} extending some $[e_i, f_{i,j}]$ or some $[A', f_{1,j}]$ and the half-primitive vectors along its sides form a half-regular triple

Lemma 4.7.1. *The junior simplex is partitioned into half-regular triangles.*

Lemma 4.7.2. *Call a chain of contractions taking a cyclic continued fraction down to [2,1,1] an MMP.*

i. Every contraction of a 1 in an MMP corresponds to a half-regular triple.

ii. For every half-regular triple of the form 1a, 2, 3 there is an MMP ending at it.

iii. Every half-regular triple appears in every MMP.

Proof. The proofs of (i) and (iii) are essentially the same as for [CR02][Lemma 2.7].

(ii) As in [CR02], if $w_2 = 2w_1 + w_3$ is a half-regular triple, then $w_1, 2w_2, w_3$ and their minuses subdivide $\mathbb{R}^2$ into 6 basic cones. The chain of vectors $f_{i,j}$ (or $2f_{i,j}$) within any cone is a non-minimal basic subdivision so contracts down.

In case (1b), v_1, v_2, v_3 are not a basis of $\mathbb{Z}^2$ because we cannot make the vector $f_{1,0}$ as a $\mathbb{Z}$-linear combination of them. In all other cases we have a vector out of A', so we can use this to build the other vectors out of A'.

(iii) The point is that if v_1, v_2, v_3 is a half-regular triple, say v_3 can be expressed as the sum of v_1 and $2v_2$ then any contraction of v_3 must involve v_1 and $2v_2$. This is because the vectors v_1 and $2v_2$ span a basic cone, and so v_3 must be expressible as a sum of a lattice vector from each of the cones $\langle v_1, v_3\rangle$ and $\langle 2v_2, v_3\rangle$. Suppose in a given MMP the first of the v_i to be affected is v_3. We know that $v_3 = u_1 + u_2$ for $u_1 \in \langle v_1, v_3\rangle$ and $u_2 \in \langle 2v_2, v_3\rangle$ half-lattice vectors, but the only possible such expression is $v_3 = v_1 + v_2$. $\square$

This proves existence and uniqueness of the partition of Lemma 4.7.1.

4.8 The four dimensional Craw-Reid knock-out contest

The fan Σ of A-Hilb$(\mathbb{C}^4)$ can be calculated by following a simple procedure:

1. Draw lines L_{ij} emanating from the corners of Δ. Record the strength a_{ij} of each L_{ij} determined by the Hirzebruch-Jung continued fraction rule.

2. Extend the lines L_{ij} until they are 'defeated' by lines L_{kl} from e_k $(i \neq k)$ according to the following rules

   - if a line from e_3 and a line from e_4 meet at a point, the line with greater strength extends but its strength decreases by 1.
   - if a line L_{Ai} from A' meets a line L_{jk} from e_j then the strength of L_{Ai} decreases by 2 if it defeated the previous line it met from e_j, otherwise it decreases by 1. If the strength of L_{Ai} decreases by 2 the strength of L_{jk} decreases by 1, otherwise it decreases by 2. See Lemma 4.6.4. The line which now has the greater strength extends.
   - if three lines meet at a point the lines $L_{3,i}$ and $L_{4,j}$ decrease in strength by 1 each due to meeting each other. Then $L_{3,i}$ decreases in strength from the meeting with $L_{A,k}$ by 1 or 2 depending on whether $L_{A,k}$ defeated the previous line it met out of e_3 or not. Similarly for e_4. The line which now has the greatest strength extends.

   If at a meeting all strengths are equal then all lines die. As in three dimensions, lines of strength 2 always die.

3. Step 2 partitions Δ into half-regular triangles. There are two cases:

   - if a half-regular triangle is equilateral with each side being s copies of a primitive vector, for some integer s, (i.e. it is actually regular) then take the regular tessellation of that triangle.
   - if a half-regular triangle has a vertex at A', then its sides are not all an integer multiple of a primitive vector. If the sides out of A' are only one copy of half-primitive vectors then no further tessellation of this triangle is necessary. If the sides are r copies of half-primitive vectors then r must be odd, say $r = 2s + 1$. Cut off the triangle closest to A', by taking a step equal to the half-primitive vector along each edge

at A'; these two points are joined by a copy of the primitive vector along the third side. Now subdivide the remaining trapezium into two equilateral triangles and a parallelogram. The parallelogram is formed by joining the mid-segment of the third line to the the base of the top triangle. The triangles on either side are equilateral so take their regular tessellation. The parallelogram is subdivided by drawing in all lines parallel to the third side, and subdividing each of the resulting parallelograms by joining opposite corners.

The result of this procedure is Σ. This may not be a crepant resolution. Dividing each parallelogram into two triangles rather than four gives a crepant resolution. The fan Σ is just a blow-up of this crepant resolution.

Example 4.8.1. The singularity $\frac{1}{57}(1,1,5,50)$ was considered in Example 4.6.6. The Hirzebruch-Jung continued fraction expansions,

$$\frac{57}{10} = [6,4,2,2] \quad \text{at } A'$$
$$\frac{57}{50} = [2,2,2,2,2,2,2,8] \quad \text{at } e_3$$
$$\frac{57}{5} = [12,2,3] \quad \text{at } e_4,$$

and Step 1 give rise to Figure 4.6. The result of extending these lines as in Step 2 is shown in Figure 4.7. The two large triangles are then triangulated as described in Step 3. Since the vectors from A' to $p_{23} = \frac{1}{57}(23,23,1,10)$ and $p_{24} = \frac{1}{57}(24,24,6,3)$ are not primitive vectors, the triangle $A'p_{23}p_{24}$ is cut off. The line segment $p_{23}p_{24}$ is joined to the line segment p_3p_4, to form a parallelogram. The parallel line segment $p_{13}p_{14}$ is inserted, and lines $p_{23}p_{14}, p_{24}p_{13}, p_{13}p_4, p_{14}p_3$ are added. Regular tessellation of the remaining triangles yields Figure 4.8. We can now form the half-regular tetrahedra by adding the vertices e_1 and e_2 to each half-regular triangle.

4.9 Invariant monomials

We move to M the lattice of invariant monomials which is dual to $L = \mathbb{Z}^4 + \frac{1}{r}(1,1,a,b)\cdot\mathbb{Z}$, and consider the dual basis of each tetrahedron in Σ.

Since our group is a subgroup of $\mathrm{SL}(4,\mathbb{C})$ the monomial $xyzt$ is invariant, and as x/y is invariant, so is x^2zt.

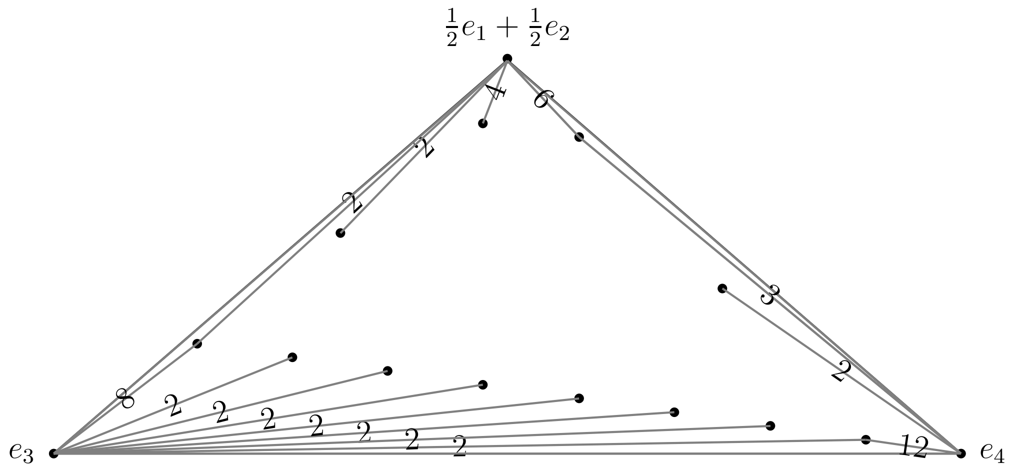

Figure 4.6: The result of step 1 for $\frac{1}{57}(1,1,5,50)$

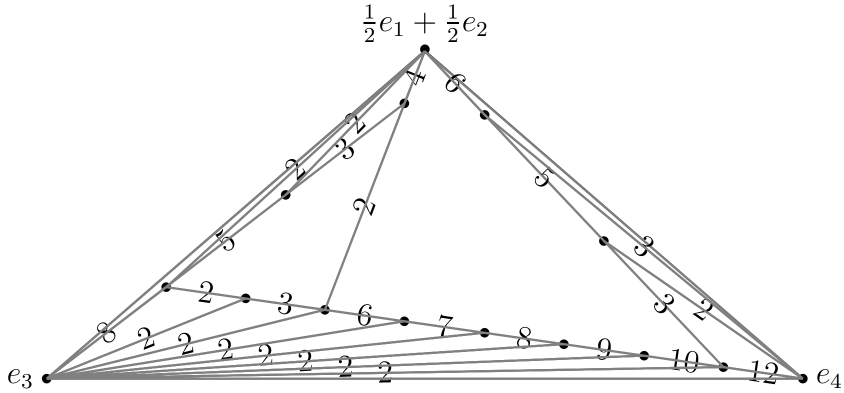

Figure 4.7: The result of step 2 for $\frac{1}{57}(1,1,5,50)$

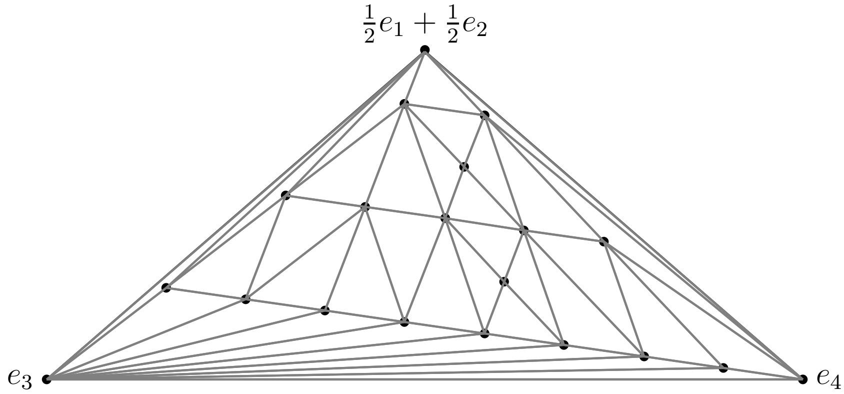

Figure 4.8: Half-regular tessellation of $A'e_3e_4$ for $\frac{1}{57}(1,1,5,50)$

To start with, although we ultimately want the dual bases of the tetrahedra, we consider the dual bases of the triangles in the triangle $A'e_3e_4$. We adopt the convention that we are on the e_2 side of this triangle (that is "below" $A'e_3e_4$). Thus $\theta = y/x$ is an element of the dual basis and every other element will be expressed in terms of x, z and t. Switching to the e_1 side (or "above") can be done by interchanging x and y in the dual basis.

Triangles containing the non-lattice point A' represent tetrahedra with both e_1 and e_2 as vertices. These triangles will be referred to as "outer" triangles, and sometimes require separate treatment.

There are five configurations of half-regular triangle are illustrated in Figure 4.9. They are

Case (a): Two lines out of A', one line out e_i, with all sides of length 1

Case (b): Two lines out of e_i, one line out of A'

Case (c): Meeting of champions; one line out of each vertex

Case (d): Two lines out of A', one line out of e_4, with sides of length greater than 1

Case (e): Two lines out of e_3, one line out of e_4

Proposition 4.9.1. *Every half-regular triangle of side r gives rise to the invariant ratios of Figure 4.9. Moreover,*

$$\textit{In case (a)} \quad d - a = e - 2c - b = f = r \tag{4.8}$$

$$\textit{In case (b)} \quad a - d = 2(b - e - c) = 2f = 2r \tag{4.9}$$

$$\textit{In case (c)} \quad a - d = 2(b - e) = 2(c - f) = 2r \tag{4.10}$$

$$\textit{In case (d)} \quad a - d = b - 2c - e = f = r = 2s + 1 \tag{4.11}$$

$$\textit{In case (e)} \quad 2(a - d) = b - e - c = 2f = 2r \tag{4.12}$$

Proposition 4.9.2. *Let l be any lattice lines of $\mathbb{Z}^2_\Delta$, and $\mathbf{m} \in M$ an invariant monomial that bases its orthogonal $l^\perp \cap M$. Then the lattice lines of $\mathbb{Z}^2_\Delta$ are orthogonal to $\mathbf{m}(x^2yz)^i$ for $i \in \mathbb{Z}$. The half-regular triangles of types 4.9(a) and 4.9(e) have side length 1.*

The regular tessellations of the half-regular triangles of Figure 4.9 are cut out

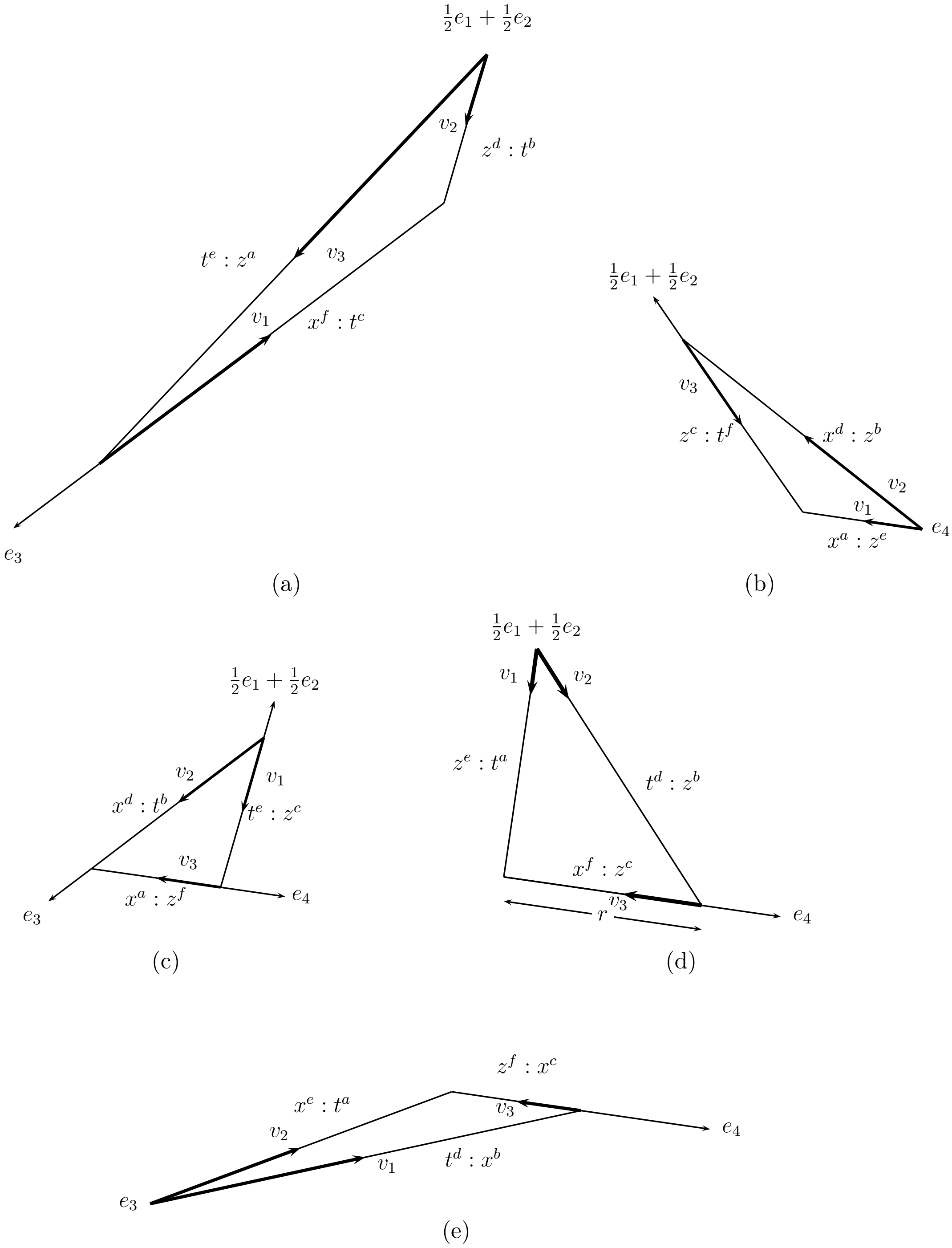

Figure 4.9: Half-regular triples versus monomials

by the ratios

$$\begin{array}{llll}
\textit{In case (a):} & x : t^c, & z^d : t^b, & t^e : z^a \\
\textit{In case (b):} & x^{a-2i} : z^{e+i}t^i, & z^{b-j} : x^{d+2j}t^j, & t^{f-k} : x^{2k}z^{c+k} \\
\textit{In case (c):} & x^{a-2i} : z^{f+i}t^i, & z^{c-j} : x^{2j}t^{e+j}, & t^{b-k} : x^{d+2k}z^k \\
\textit{In case (e):} & x^{b-2i} : t^{d+i}z^i, & z^{f-j} : x^{c+2j}t^j, & t^{a-k} : x^{e+2k}z^k
\end{array}$$

Corollary 4.9.3. *The regular tessellations of regular triangles of Figure 4.9(d) are cut out by the ratios:*

$$\begin{array}{rlll}
\textit{Left triangle} & x^{f-2i} : z^{c+i}t^i, & z^{e+c+s-j} : x^{f-2s+2j}t^{a+s+j}, & t^{a-k} : x^{2k}z^{e+k} \\
\textit{Right triangle} & x^{f-2i} : z^{c+i}t^i, & z^{b-j} : x^{2j}t^{d+j}, & t^{d+s-k} : x^{f-2s+2k}z^{b-c-s+k}
\end{array}$$

The tessellations of the alleys of parallelograms are cut out by the ratios:

$$x^{2i}z^{e+i} : t^{a-i}, \quad z^{b-j} : x^{2j}t^{d+j}$$

and one of

$$x^{f-2s}t^{a-s} : z^{e+c+s}, z^{c+k}t^k : x^{f-2k}, x^{f-2s}z^{b-c-s} : t^{d+s}$$

Proof of 4.9.1 and 4.9.3. Case (d)

$$\begin{aligned}
v_1 &\sim \left(\frac{-a-e}{2}, \frac{-a-e}{2}, a, e\right) \\
v_2 &\sim \left(\frac{-d-b}{2}, \frac{-d-b}{2}, d, b\right) \\
v_3 &\sim (c, c, f, -2c-f).
\end{aligned} \tag{4.13}$$

Claim, that

$$\frac{1}{ab-de} = \frac{1}{2ac+af+ef} = \frac{1}{2dc+df+bf}$$

where the denominators are the 2×2 minors of the array given by (4.13).

Let

$$\xi = \frac{z^c}{x^f}, \quad \eta = \frac{t^d}{z^b}, \quad \zeta = \frac{z^e}{t^a}.$$

Then

$$\begin{aligned}
v_1(\xi) = v_2(\xi) = v_3(\zeta) = 1 \\
v_1(\eta) = v_2(\zeta) = v_3(\eta) = -1.
\end{aligned}$$

In case (d), we have $v_2 = v_1 + v_3$. Comparing coefficients we get $d = a - f$ and $b = e - 2c - f$, which are the first two equalities in (4.8).

Now,

$$\begin{aligned} A' &+ fv_1 \\ &= \frac{1}{2ac+af+ef}\left(\frac{2ac+af+ef}{2} - \frac{af+ef}{2}, \frac{2ac+af+ef}{2} - \frac{af+ef}{2}, af, ef\right) \\ &= \frac{1}{2ac+af+ef}(ac, ac, af, ef). \end{aligned}$$

The first three entries ac, ac, af are proportional to c, c, f so lie on the third side of R. Therefore $r = f$.

For Corollary 4.9.3, we obtain all the ratios by taking $\xi(x^2zt)^i, \eta(x^2zt)^i, \zeta(x^2zt)^i$ and $\xi\zeta(x^2zt)^j, \eta\zeta(x^2zt)^j$. The proofs of the other cases are similar. □

Let R be a regular triangle of side r. Every basic triangle is one of two types. We use the "up" and "down" triangle terminology from [CR02]:

"up": For $i, j, k \geq 0$ with $i + j + k = r - 1$, push the three sides of R inwards by i, j and k lattice steps respectively to give a basic triangle T. The sides of T are parallel to the sides of R, so that T is a scaled down version of R.

"down": For $i, j, k \geq 0$ with $i+j+k = r+1$, push the three sides of R inwards by i, j and k lattice steps. The resulting triangle, T, is a scaled down version of R which has been inverted.

Corollary 4.9.4. *The dual bases of basic up triangles are given by:*

$$\begin{array}{llll} \textit{In case (a):} & \xi = x/t^c, & \eta = z^d/t^b, & \zeta = t^e/z^a \\ \textit{In case (b):} & \xi = x^{a-2i}/z^{e+i}t^i, & \eta = z^{b-j}/x^{d+2j}t^j, & \zeta = t^{f-k}/x^{2k}z^{c+k} \\ \textit{In case (c):} & \xi = x^{a-2i}/z^{f+i}t^i, & \eta = z^{c-j}/t^{e+j}x^{2j}, & \zeta = t^{b-k}/x^{d+2k}z^k \\ \textit{In case (e):} & \xi = x^{b-2i}/t^{d+i}z^i, & \eta = z^{f-j}/x^{c+2j}t^j, & \zeta = t^{a-k}/x^{e+2k}z^k \end{array} \tag{4.14}$$

with $0 \leq i, j, k < r$ *and* $i + j + k = r - 1$.

The dual bases of down triangles are given by:

$$\begin{array}{llll} \textit{In case (b):} & \lambda = z^{e+i}t^i/x^{a-2i}, & \mu = x^{d+2j}t^j/z^{b-j}, & \nu = x^{2k}z^{c+k}/t^{f-k} \\ \textit{In case (c):} & \lambda = z^{f+i}t^i/x^{a-2i}, & \mu = t^{e+j}x^{2j}/z^{c-j}, & \nu = x^{d+2k}z^k/t^{b-k} \\ \textit{In case (e):} & \lambda = t^{d+i}z^i/x^{b-2i}, & \mu = x^{c+2j}t^j/z^{f-j}, & \nu = x^{e+2k}z^k/t^{a-k} \end{array}$$

with $0 \le i, j, k < r$ *and* $i + j + k = r + 1$.

Corollary 4.9.3 gives the ratios which subdivide the half-regular triangle into two regular triangles separated by an alley of parallelograms. We will refer to the triangle to the left of this alley as a "left" triangle and the triangle to the right as a "right" triangle. Each parallelogram is divided into four pieces. These are described as "left", "right", "up" and "down". The "top" triangle is the triangle above the alley with a vertex at A'.

Corollary 4.9.5. *The dual bases for basic triangles of type (d) are given by:*

Left Up

$$\xi = x^{f-2i}/z^{c+i}t^i, \quad \eta = z^{e+c+s-j}/x^{f-2s+2j}t^{a-s+j}, \quad \zeta = t^{a-k}/x^{2k}z^{e+k} \tag{4.15}$$

with $0 \le i, j, k < s$ *and* $i + j + k = s - 1$

Left Down

$$\lambda = z^{c+i}t^i/x^{f-2i}, \quad \mu = x^{f-2s+2j}t^{a-s+j}/z^{e+c+s-j}, \quad \nu = x^{2k}z^{e+k}/t^{a-k}$$

with $0 \le i, j, k < s$ *and* $i + j + k = s + 1$

Right Up

$$\xi = x^{f-2i}/z^{c+i}t^i, \quad \eta = z^{b-j}/x^{2j}t^{d+j}, \quad \zeta = t^{d+s-k}/x^{f-2s+2k}z^{b-c-s+k}$$

with $0 \le i, j, k < s$ *and* $i + j + k = s - 1$

Right Down

$$\lambda = z^{c+i}t^i/x^{f-2i}, \quad \mu = x^{2j}t^{d+j}/z^{b-j}, \quad \nu = x^{f-2s+2k}z^{b-c-s+k}/t^{d+s-k}$$

with $0 \le i, j, k < s$ *and* $i + j + k = s + 1$

Parallelograms

Left

$$\mu = x^{f-2s}t^{a-s}/z^{e+c+s}, \quad \nu = x^{2i}z^{e+i}/t^{a-i}, \quad \eta = z^{b-j}/x^{2j}t^{d+j} \tag{4.16}$$

with $i = j$, $1 \le i, j \le s$.

Up

$$\xi = x^{f-2i}/z^{c+i}t^i, \quad \eta = z^{b-j}/x^{2j}t^{d+j}, \quad \zeta = t^{a-k}/x^{2k}z^{e+k},$$

with $2i + j + k = r - 1$, $0 \le i \le s - 1$ *and* $1 \le j = k \le s$.

Down

$$\lambda = z^{c+i}t^i/x^{f-2i}, \quad \nu = x^{2k}z^{e+k}/t^{a-k}, \quad \mu = x^{2j}t^{d+j}/z^{b-j}$$

with $2i+j+k = r+1$, $1 \leq i,j,k \leq s$ *and* $j = k$.

Right

$$\nu = x^{f-2s}z^{b-c-s}/t^{d+s}, \quad \zeta = t^{a-i}/x^{2i}z^{e+i}, \quad \mu = x^{2j}t^{d+j}/z^{b-j}$$

with $i = j$, $1 \leq i,j \leq s$.

Top triangle

$$\xi = x^{f-2s}/z^{c+s}t^s, \quad \eta = z^b/t^d, \quad \zeta = t^a/z^e. \tag{4.17}$$

Example 4.9.6. In Example 4.6.5 we found the partition of $A'e_3e_4$ into half-regular triangles. It is not hard to see that the only half-regular triangle which does not have side 1 is given by the triple $f_{4,1} = f_{A,2} - f_{A,1}$. The tetrahedron with this triangle as base and additional vertex e_2 is cut out by the ratios $z : t^4, z^6 : t, y : x$ and $x^3 : z$. The ratios cutting out the interior lines are given by

$$x^{2i}z^{1+i} : t^{4-i}, \quad x^{2j}t^{1+j} : z^{6-j}, \quad x^{3-2k} : z^{1+k}t^{2k}, \quad z^3 : xt^3, \quad x : z^2t, \quad xz^4 : t^2.$$

for integers $0 \leq i,j,k \leq 1$. The ratios for the whole triangle $A'e_3e_4$ (on the e_2 side) are given in Figure 4.10.

4.10 A-Hilb$(\mathbb{C}^4)$ and A-clusters

Theorem 4.10.1. *Let* $A = \frac{1}{r}(1,1,a,b)$ *be a finite subgroup of* $\mathrm{SL}(4,\mathbb{C})$ *with* a, b *not both odd and which satisfies Condition 4.5.1. For every A-cluster Z, generators of the ideal $\mathcal{I}_Z$ can be chosen as a system of equations. Throughout* $a,b,c,d,e,f,m,n \geq 0$ *are integers and* $\xi,\eta,\zeta,\lambda,\mu,\nu,\pi \in \mathbb{C}$ *are constants. There are six cases:*

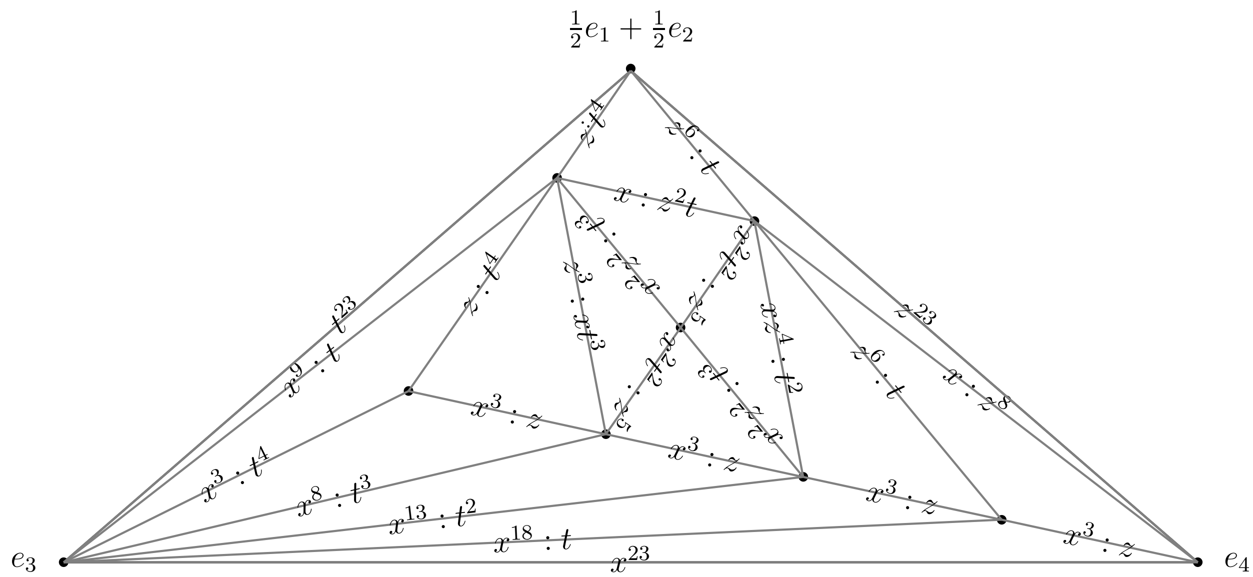

Figure 4.10: The dual basis for $\frac{1}{23}(1, 1, 3, 18)$

1. *(I) "Outer triangles":*

$$x = \xi z^b t^f, \quad y = \lambda z^b t^f,$$
$$z^{m+1} = \eta t^c,$$
$$t^{n+1} = \zeta z^e,$$
$$xyzt = \pi,$$

with $\xi, \eta, \zeta, \lambda, \pi$ satisfying

$$\xi\eta\zeta\lambda = \pi,$$

(II) Moreover the following hold:

$$n = c + 2f, \quad m = 2b + e.$$

2. *(I) Left parallelogram triangles:*

$$\begin{aligned} x^{l+1} &= \xi z^b t^f, & z^{b+1}t^{f+1} &= \lambda x^{l-1}, \\ z^{m+1} &= \eta x^d t^c, & xt^{f+c+1} &= \mu z^{b+e+1}, \\ t^{n+1} &= \zeta x^a z^e, & x^{a+2}z^{e+1} &= \nu t^n, \\ xyzt &= \pi, & y &= \theta x, \end{aligned} \tag{4.18}$$

with $\xi, \zeta, \lambda, \nu, \pi$ *satisfying*

$$\xi\lambda\theta = \zeta\nu\theta = \pi,$$

(II) Moreover the following hold:

$$\begin{array}{l} \mu^2\nu\eta\theta = \pi, \quad \xi = \mu\nu, \quad \zeta = \mu^2\eta, \quad \lambda = \eta\mu, \\ l = d = a + 2, \quad m = 2b + e + 1, \quad n = 2f + c + 1. \end{array} \tag{4.19}$$

3. *(I) Down parallelogram triangles:*

$$\begin{aligned} x^{l+1} &= \xi z^b t^f, & z^{b+1}t^{f+1} &= \lambda x^{l-1}, \\ z^{m+1} &= \eta x^d t^c, & x^{d+2}t^{c+1} &= \mu z^m, \\ t^{n+1} &= \zeta x^a z^e, & x^{a+2}z^{e+1} &= \nu t^n, \\ xyzt &= \pi, & y &= \theta x, \end{aligned}$$

with $\eta, \zeta, \mu, \nu, \pi$ *satisfying*

$$\eta\mu\theta = \zeta\nu\theta = \pi,$$

(II) Moreover the following hold:

$$\begin{array}{l} \lambda^2\mu\nu\theta = \pi, \quad \xi = \lambda\mu\nu, \quad \zeta = \lambda^2\mu, \quad \eta = \lambda^2\nu, \\ 2l = a + d + 3, \quad m = 2b + e + 2, \quad n = 2f + c + 1. \end{array}$$

4. *(I) Up parallelogram triangles:*

$$\begin{aligned} x^{l+1} &= \xi z^b t^f, & z^{b+1}t^{f+1} &= \lambda x^{l-1}, \\ z^{m+1} &= \eta x^d, & xt^{f+c+1} &= \mu z^{b+e}, \\ t^{n+1} &= \zeta x^a, & xz^{b+e+1} &= \nu t^{f+c}, \\ xyzt &= \pi, & y &= \theta x, \end{aligned}$$

with $\xi, \zeta, \lambda, \nu, \pi$ *satisfying*

$$\xi\lambda\theta = \zeta\nu\theta = \pi.$$

(II) Moreover the following hold:

$$\begin{aligned}&\xi^2\eta\zeta\theta=\pi, \quad \lambda=\xi\eta\zeta, \quad \mu=\xi\zeta, \quad \nu=\xi\eta,\\ &2l=a+d, \quad m=2b+e, \quad n=2f+c.\end{aligned}$$

5. *(I) Right parallelogram triangles:*

$$\begin{aligned} x^{l+1}&=\xi z^b t^f, & z^{b+1}t^{f+1}&=\lambda x^{l-1},\\ z^{m+1}&=\eta x^d t^c, & x^{d+2}t^{c+1}&=\mu z^m,\\ t^{n+1}&=\zeta x^a z^e, & xz^{b+e+1}&=\nu t^{c+f+1},\\ xyzt&=\pi, & y&=\theta x,\end{aligned}$$

with ξ,η,λ,μ,π *satisfying*

$$\xi\lambda\theta=\eta\mu\theta=\pi,$$

(II) Moreover the following hold:

$$\begin{aligned}&\lambda\mu\nu\theta=\pi, \quad \xi=\mu\nu, \eta=\lambda\nu, \lambda=\nu\zeta\\ &l=a=d+2, \quad m=2b+e+1, \quad n=2f+c+1.\end{aligned}$$

6. *(I) All other interior triangles:*

$$\begin{aligned} x^{l+1}&=\xi z^b t^f, & z^{b+1}t^{f+1}&=\lambda x^{l-1},\\ z^{m+1}&=\eta x^d t^c, & x^{d+2}t^{c+1}&=\mu z^m,\\ t^{n+1}&=\zeta x^a z^e, & x^{a+2}z^{e+1}&=\nu t^n,\\ xyzt&=\pi, & y&=\theta x,\end{aligned} \tag{4.20}$$

with $\xi,\eta,\zeta,\lambda,\mu,\nu,\pi$ *satisfying*

$$\xi\lambda\theta=\eta\mu\theta=\zeta\nu\theta=\pi \tag{4.21}$$

(II) Moreover, exactly one of the following hold:

$$\text{“up”}\begin{cases} \xi\eta\zeta\theta = \pi, \quad \lambda = \eta\zeta, \quad \mu = \xi\zeta, \quad \nu = \xi\eta, \\ l = a+d+1, \quad m = e+b, \quad n = f+c \end{cases}$$

$$\text{“down”}\begin{cases} \lambda\mu\nu\theta = \pi, \quad \xi = \nu\mu, \quad \zeta = \lambda\mu, \quad \eta = \lambda\nu \\ l = a+d+3, \quad m = e+b, \quad n = f+c \end{cases}$$

A *basic monomial* $\mathbf{m}$ is the nonzero image in $\mathcal{O}_Z = k[x,y,z,t]/I_Z$ of a monomial which is not an invariant monomial. Basic monomials cannot be a multiple of $xyzt$, x^2zt or y^zt, so must be a multiple of at most three of x,y,z,t. Since x and y are in the same eigenspace they cannot be a multiple of xy either.

The following lemma is required for the proof of Theorem 4.10.1.

Lemma 4.10.2. *[CR02] Let x^r be the first power of x that is A-invariant. Then there is (at least) one $l \in [0, r-1]$ such that $1, x, x^2, \dots, x^l \in \mathcal{O}_Z$ are basic monomials and x^{l+1} is a multiple of some basic monomial z^bt^f in the same eigenspace, say $x^{l+1} = \xi z^bt^f$ for some $\xi \in \mathbb{C}$.*

The proof of Lemma 4.10.2 is as in [CR02], but with the additional observation that for $l > 0$ the monomial x^{l+1} cannot be expressed as a multiple of y since $y = \theta x$, so $y = 0 \in \mathcal{O}_Z$. If $l = 0$ then either $x = \theta y$ or $x = \xi z^bt^f$.

Proof of 4.10.1. We have $xyzt = \pi$ since $A \in \mathrm{SL}(4,\mathbb{C})$ for $\pi \in \mathbb{C}$. Also since $A = \frac{1}{r}(1,1,a,b)$ we must have x and y in the same eigenspace, so if $x, y \neq 0 \in \mathcal{O}_Z$ then there is a relation $y = \theta x$ for some $\theta \in \mathbb{C}$. If $x, y = 0 \in \mathcal{O}_Z$ then $x = \xi z^bt^f$ and $y = \lambda z^bt^f$.

By Lemma 4.10.2, x^{l+1} and y^bt^f belong to a common eigenspace and, since x^2zt is invariant, x^{l-1} and $z^{b+1}t^{f+1}$ also belong to a common eigenspace. Now x^{l-1} is basic so this eigenspace is based by x^{l-1} which gives the relation $z^{b+1}t^{f+1} = \lambda x^{l-1}$.

To see the equation $\xi\lambda\theta = \pi$ first note the syzygy $(xyzt - \pi) - (xzt)(y - \theta x) = \theta x^2zt - \pi = h_1$. Now using this and the relations $h_2 = x^{l+1} - \xi z^bt^f$ and $h_3 = z^{b+1}t^{f+1} - \lambda x^{l-1}$ in the syzygy $\lambda\theta h_2 + \theta x^2 h_3 - z^bt^f h_1$ gives $\xi\lambda\theta z^bt^f = \pi z^bt^f$ as required.

The relations are in pairs $x^{l+1} \mapsto z^bt^f$, $z^{b+1}t^{f+1} \mapsto x^{l-1}$. The first relation reduces the pure powers of x higher than l. Suppose there is another relation of the form $x^\alpha z^\epsilon \mapsto \mathbf{m}$. If $\mathbf{m}$ involves x, y or z this relation would be a multiple of a simpler relation. However, if $\mathbf{m} = t^\gamma$ is a pure power of t, the above argument shows that this relation is paired with $t^\gamma \mapsto x^{\alpha-1}z^{\epsilon-1}$ which contradicts our choice of n (in the exponent of t^{n+1}).

For the left parallelogram triangles the relations involving η, μ, ν, θ generate the others. In this situation we prove that there are a pair of relations of the form

$$x^{l+1} = \xi z^b t^f, \quad z^{b+1}t^{f+1} = \lambda x^{l-1},$$

such that $\xi = \mu\eta$ and $\lambda = \eta\mu$.

Consider the left parallelogram triangles. We have

$$x^{a+3}t^{f+c+1} \mapsto \mu x^{a+2}z^{b+e+1} \mapsto \mu\nu z^b t^n.$$

Then $x^{a+3}t^{f+c+1}$ and $z^b t^n$ are in the same eigenspace, so if $n \geq f+c+1$ there is a relation

$$x^{a+3} = \mu\nu z^b t^{n-f-c-1}$$

and since this is basic, the argument above means that there is a unique equation of this form. Thus $\xi = \mu\nu$, $l = a+2$ and $n = 2f+c+1$. Also

$$z^{m+1}t^{f+1} \mapsto \eta x^d t^{f+c+1} \mapsto \eta\mu x^{d-1}z^{b+e+1},$$

so if $b+e \leq m$ then we have the relation

$$z^{m-b-e}t^{f+1} = \eta\mu x^{d-1}.$$

This is again basic, so must equal $z^{b+1}t^{n-f-c} = \lambda x^{a+1}$. Thus $\lambda = \eta\mu$ and $l = d = a+2$ and $m = 2b+e+1$.

If $n < f+c+1$ we have

$$x^{a+3}t^{f+c+1-n} = \mu\nu z^b$$

which contradicts the choice of m. In the same way $m+1 < b+e+1$ is not allowed.

A similar argument proves the relations of (II) for the other interior triangles.

$\square$

Theorem 4.10.3. *Let Σ denote the toric fan determined by the tessellation described in Section 4.8 of all half-regular triangles in the junior simplex Σ. The associated toric variety is the A-Hilbert scheme A-Hilb($\mathbb{C}^3$).*

Proof. We do this in a few cases. The proofs for the other cases are similar.
Case (c) "up": Substituting d, e, f using (4.10), and replacing a, b, c with A, B, C respectively in (4.14) gives

$$x^{A-2i} = \xi z^{C-r+i} t^i, \quad z^{C-j} = \eta t^{B-r+j} x^{2j}, \quad t^{B-k} = \zeta x^{A-2r-2k} z^k.$$

Let $y = \theta x$. Then we see that

$$\begin{aligned} x^{A-2i} &= \xi z^{C-r+i} t^i, & z^{C-j-k} t^{r-j-k} &= \eta\zeta x^{A+2j+2k-2r}, \\ z^{C-j} &= \eta t^{B-r+j} x^{2j}, & x^{2r-2i-2k} t^{B-i-k} &= \xi\zeta z^{C-r+i+k}, \\ t^{B-k} &= \zeta x^{A-2r+2k} z^k, & x^{A-2i-2j} z^{r-i-j} &= \xi\eta t^{B-r+i+j}, \\ xyzt &= \xi\eta\zeta\theta, & y &= \theta x. \end{aligned}$$

which are exactly the "up" version of equations (4.20), since $i + j + k = r - 1$, with $l = A - 2i - 1, b = C - r + i, f = i$ etc.
Left "up" triangles: Replacing a, c, e, f with A, C, E, F respectively in (4.15), and letting $y = \theta x$ gives

$$\begin{aligned} x^{F-2i} &= \xi z^{C+i} t^i, & z^{C+i+1} t^{i+1} &= \eta\zeta x^{F+2i-2}, \\ z^{E+C+s-j} &= \eta x^{F-2s+2j} t^{A-s+j}, & x^{F-2i-2k} t^{A-k-i} &= \xi\zeta z^{C+E+k+i}, \\ t^{A-k} &= \zeta x^{2k} z^{E+k}, & x^{2+2k} z^{E+k+1} &= \xi\eta t^{A-k-1}, \\ xyzt &= \xi\eta\zeta\theta, & y &= \theta x. \end{aligned}$$

which are exactly the "up" version of equations (4.20) with $l = F - 2i, b = C + i, f = i$ etc.
Left parallelogram triangles: Equation (4.16) with d, e, f substituted using (4.11), and replacing a, b, c with A, B, C respectively gives

$$\begin{aligned} x^{1+2i} &= \mu\nu t^{s-i} z^{C+s-i}, & z^{C+s+1-j} t^{s+1-j} &= \mu\eta x^{2j-1}, \\ z^{B-j} &= \eta x^{2j} t^{A-2s-1+j}, & xt^{A-s} &= \mu z^{B-C-s-1}, \\ t^{A+1-j} &= \mu^2 \eta x^{2+2j} z^{B-2C-s-2+j}, & tx^{2i} z^{B-2C-2s-1+i} &= \nu t^{A-i}, \\ xyzt &= \mu^2\eta\nu\theta, & y &= \theta x. \end{aligned}$$

which are exactly the equations (4.18) with $l = 1 + 2i$, $b = C + s - i$, $f = s - i$ etc.

We now prove the converse.

All other interior triangles: In the "up" case these are generated by the equations

$$x^{a+d+2} = \xi z^b t^f, \quad z^{e+b+1} = \eta x^d t^c, \quad t^{f+c+1} = \zeta x^a z^e, \quad y = \theta x,$$

Let $b = C + i$, $f = i$, $d = F - 2s + 2j$, $c = A - s + j$, $a = 2k$, $e = E + k$. Then we have

$$\begin{aligned} x^{F-2s+2j+2k+2} &= \xi z^{C+i} t^i, & z^{C+E+i+k+1} &= \eta x^{F-2s-2j} t^{A-s+j}, \\ t^{A-s+i+j+1} &= \zeta x^{2k} z^{E+k}, & y &= \theta x, \end{aligned}$$

which are the equations of (4.15) if $i + j + k = s - 1$.
Left parallelogram triangles: These are generated by the equations

$$z^{2b+e+2} = \eta x^d t^c, \quad x t^{2f+c+1} = \mu z^{b+e+1}, \quad x^{a+2} z^{e+1} = \nu t^{2f+c+1}, \quad y = \theta x,$$

Let $a = 2i - 2$, $b = C + s - i$, $c = D + j$, $e = E + i - 1$, $f = A - D - s - j - 1$. Then we have

$$\begin{aligned} z^{2C+E+2s-i+1} &= \eta x^{2i} t^{D+j}, & x t^{A-s} &= \mu z^{C+E+s}, \\ x^{2i} z^{E+i} &= \nu t^{2A-D-2s-j-1}, & y &= \theta x, \end{aligned}$$

which are the equations of (4.16) if $i = j$, $B - 2C - E = 2s + 1$, $A - D = 2s + 1$ and $F = 2s + 1$.
"Outer triangles": These are generated by the equations

$$x = \xi z^b t^f, \quad y = \lambda z^b t^f, \quad z^{2b+e+1} = \eta t^c, \quad t^{c+2f+1} = \zeta z^e. \tag{4.22}$$

Let $b = C + s$, $f = s$, $c = D$, $e = E$. Then we have

$$\begin{aligned} x &= \xi z^{C+s} t^s, & y &= \lambda z^{C+s} t^s, \\ z^{2C+E+2s+1} &= \eta t^D, & t^{D+2s+1} &= \zeta z^E. \end{aligned}$$

which are the equations of (4.9.5) if $A - D = 2s + 1$ and $B - 2C - E = 2s + 1$. $\square$

We can now prove

Theorem 4.5.3. *There exists a crepant resolution if and only Condition 4.5.1 is satisfied.*

Proof. We have already proved that the condition is necessary in section 4.5. To prove the converse we calculate the toric fan, Σ, as described above. Contracting Σ at the age two points (i.e. the crossing points of the diagonals of the parallelograms) gives a crepant resolution of $\mathbb{C}^4/A$. $\square$

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